

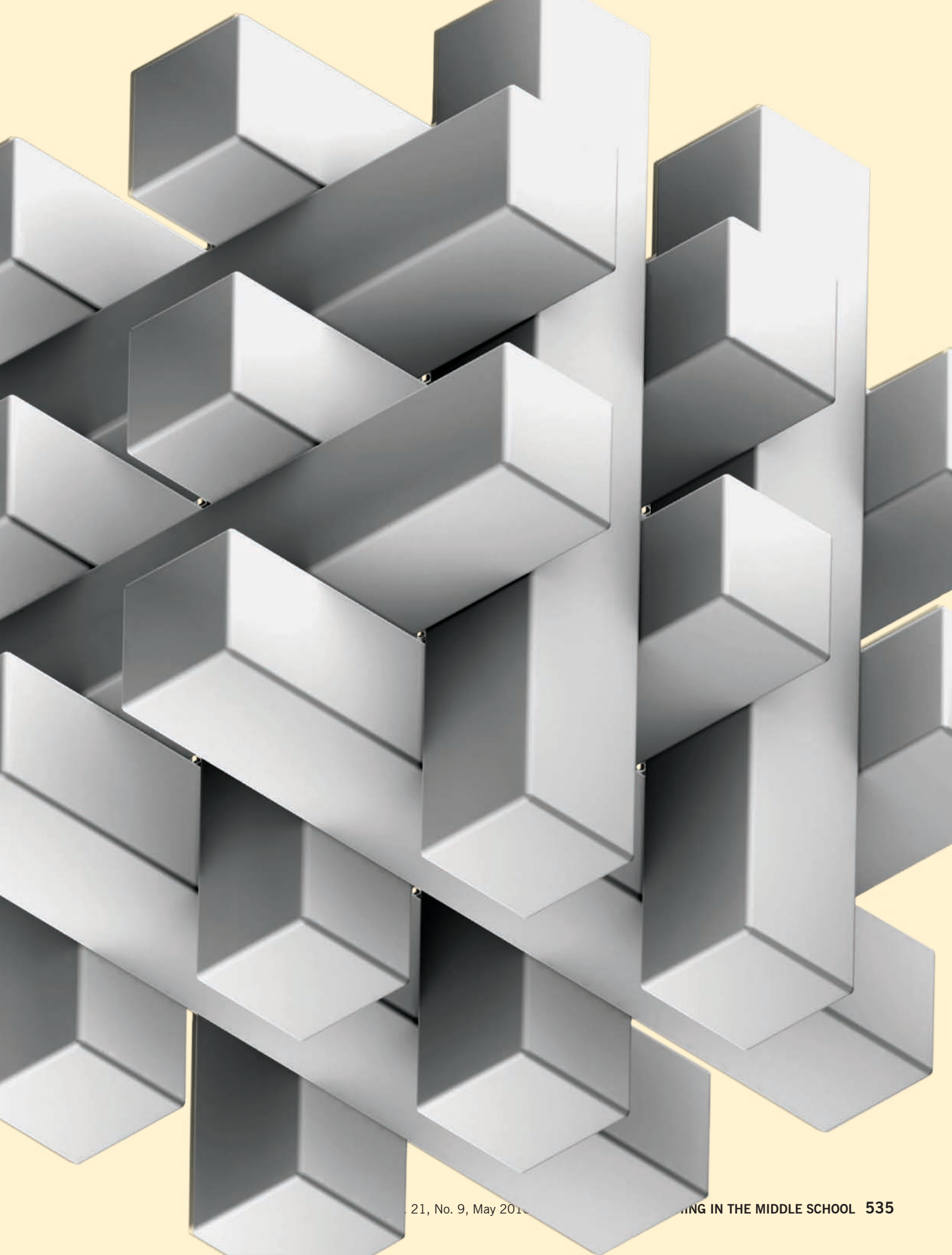
# Expanding Geometry Understanding with 3D Printing

*Want an affordable, powerful tool that can be used to teach such topics as solids, volume, and cross sections? Look into a 3D printer.*

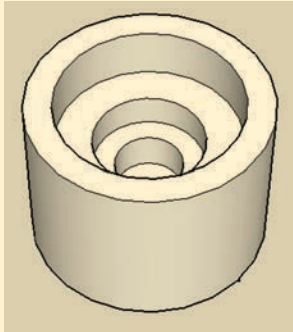
Jill A. Cochran, Zane Cochran, Kendra Laney, and Mandi Dean

**W**ith the rise of personal desktop 3D printing, a wide spectrum of educational opportunities has become available for educators to leverage this technology in their classrooms. Until recently, the ability to create physical 3D models was well beyond the scope, skill, and budget of many schools. However, since desktop 3D printers have become readily available and increasingly affordable, there has been an explosion in the varieties and quality of 3D printers and software. An extensive list and comparison of a range of 3D printers can be found in Dougherty's review (2013), including the printer we use most often (UP Mini), which can be purchased for less than \$600.

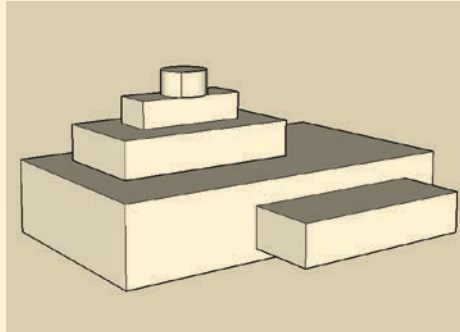
For the past few years, a physical computing lab at Berry College, Hack-Berry Lab, has had the opportunity to experiment and share this technology with students, ranging from kindergartners to undergraduates, through lab visits, tours, and even formal classes. As we have engaged these diverse groups, various elements of the 3D printing process have resonated with students, allowing them to understand traditional concepts in new and tangible ways. With this in mind, we began developing a curriculum that could



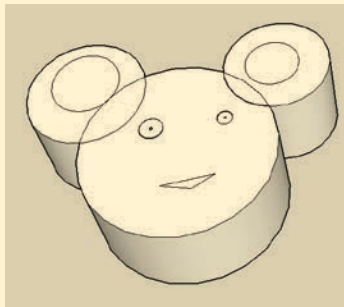
**Fig. 1** These designs were created by fourth graders (designs a–d); seventh graders followed instructions (e) to create a design (f).



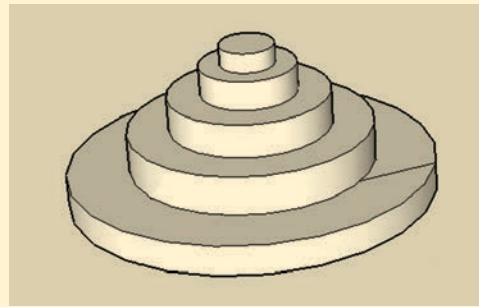
(a)



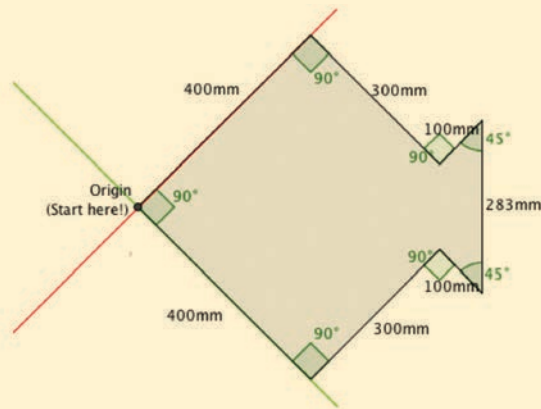
(b)



(c)

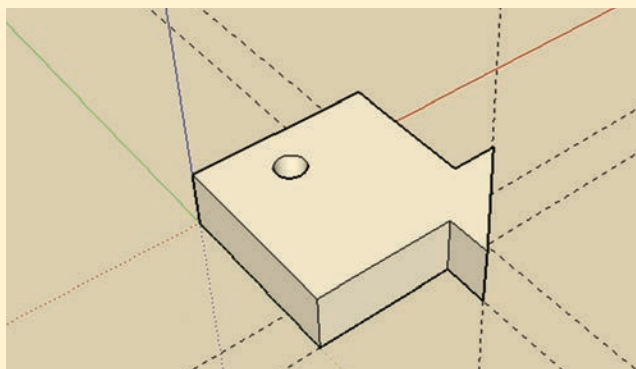


(d)



\*Follow this picture to design your fish in SketchUp.

(e)



(f)

leverage the physical, methodical, and observable nature of 3D printing and apply it to geometric concepts that are abstract and oftentimes difficult for students to grasp.

Our goal to develop geometry curriculum for students using 3D printing as an instructional tool required two initial steps. First, we began exploring the appropriateness of introducing this technology to young students. Because of the technical aspects of the process as well as some of the expectations in manipulating and creating models in 3D CAD (computer aided design) software, it became important for us to explore the landscape of available software and determine the ages of students who would be able to grasp the concepts presented and excel at applying the technology to their learning.

Second, creating lessons that addressed curricular goals and standards played an important role in our planning and execution. Although many schools and libraries have been quick to acquire 3D printers, many have done so without a clear vision of what role these tools will play. Often, such tools as this take an informal role in education and are relegated to extracurricular activity instead of regular classroom use (Magloire and Aly 2013). Even when the 3D printers have been used for classroom teaching, they are usually employed as a teacher resource, such as to print manipulatives like those available from MakerBot's Thingiverse website, rather than as a tool for students to use. One guiding principle in our activities has been deploying 3D printers in the classroom with students and allowing them to inform formal instruction.

With these considerations in mind, we then set about implementing a series of classroom-based lessons with fourth-grade and seventh-grade students who were studying geometry.





Another argument for a 3D printer is the tactile experience that results from producing such varied objects as these.



One student's design was analyzed for 3D viability.

While working in close connection with the classroom teachers, we were able to meet with each class once a week over three weeks to engage the students in creative exploration, design, and application of 3D printing as it pertained to the specific geometry topics within their regular curriculum.

### INTRODUCTORY LESSON

As we began working with the students, we knew we would need to introduce them to design software and to the idea of 3D printing. Thus, the first lesson for both classes was similar. We showed the students the

3D printer and used a webcam mounted inside it to project a live video stream, thus allowing all students to clearly see without crowding around one machine. We discussed how a 3D printer works, comparing it with a very precise glue gun that ejects a small amount of plastic at a time while a platform moves left and right, front and back, and up and down, according to the computer instructions. Using video from an object in production, we explained how designs are printed one layer at a time.

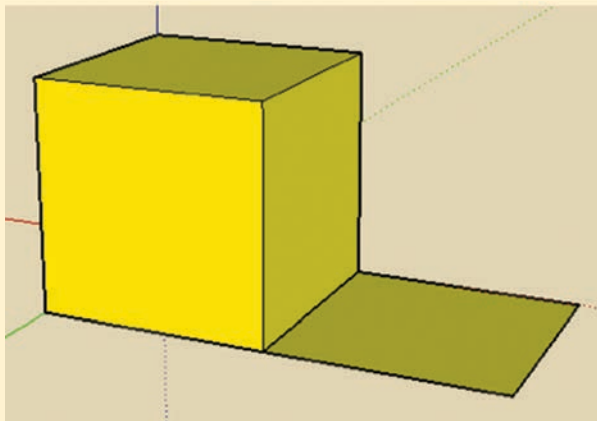
With this basic knowledge, we introduced the students to SketchUp Make, a free design software that

we anticipated would be the most appropriate for the given age levels while offering appropriate challenges (SketchUp 2015). The first mathematical conversation addressed the given lines on the screen, which were the  $x$ -,  $y$ -, and  $z$ -axes. The students' initial guesses were that they were "guidelines" or "borders," which provided opportunities to connect this information to coordinates and graphing concepts that they had experienced.

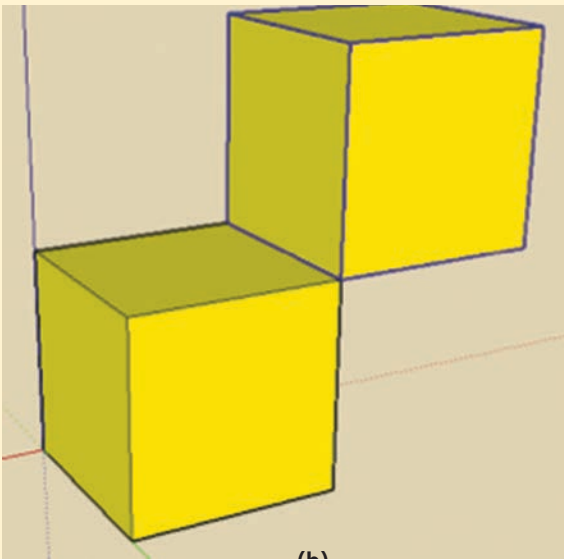
Designs in SketchUp often begin with two-dimensional shapes that are then extruded into three dimensions. The students first used the rectangle tool. The fourth graders made rectangles freehand, whereas the seventh graders and some fourth graders were pushed to enter precise measurements to ensure that they created a 10 mm  $\times$  10 mm rectangle, or square. The robustness of the software allowed us to be able to differentiate the instruction on the basis of students' levels of understanding. In addition to measurement concepts, we challenged some students to draw different two-dimensional shapes and anticipate what they would look like in three dimensions. We introduced the students to the push and pull tool, which extrudes a two-dimensional object into three dimensions, such as a circle to a cylinder or a square to a rectangular prism. Each time this tool was taught, there was great audience reaction. The exclamations gave us confidence that there was something truly significant about making the connection between two- and three-dimensional objects and that being able to create three-dimensional objects was a powerful tool.

As we introduced other tools in SketchUp to the students, each class became more confident; by the end of the first lesson, all students had created an object to print. To practice with the tools, the fourth graders created their

**Fig. 2** Students learned that various designs that did not contain volume would not print.



(a)



(b)

own designs while the seventh graders followed a set of directions, including length and angle measurements. **Figure 1** (see p. 536) contains some of the fourth-grade and seventh-grade designs. Because of time constraints, we completed the printing outside of class, so that we could print multiple student designs at one time and follow up with the students during the next class session. Within the context of 3D printing, students' designs like those in **figure 2** became powerful examples. A few designs did not print. As we showed the fourth graders cer-

tain designs, we asked them to comment about whether an object would print or not. After observing the first design in **figure 2a**, several students commented that part of the design was "flat and not three-dimensional" and that "the printer wouldn't have anything to print in that part," correctly concluding that the 3D printer would have nothing to print because there was no volume. **Figure 2b** consisted of two cubes connected by a one-dimensional line. Students recognized that this line was not enough to hold the two cubes together. Some

students guessed that the printer would not attempt to print it or that "it might try printing, but it would be very weak and break off easily" because the connection between the parts, the one-dimensional line, had no volume. For both of these powerful examples of dimension, the students initially showed good intuition in the context of whether something would print. Then we reinforced their observations by helping them associate the "weak" and "flat" parts of the designs with one and two-dimensional objects that lack volume. These ideas about dimension, 3D printing, and solid objects provided a foundation of geometry knowledge on which we could build in the following lessons. We will share two of them in this article.

### VOLUME LESSON

In a lesson from our three-week pilot study with a fourth-grade class, we wanted to use both the 3D printing process and SketchUp software to help students increase their conceptual understanding of volume while developing their own strategies and general formulas for finding the volume of prisms with different bases.

In this lesson, we asked students to share their current ideas of volume. One student described volume as the "size of space or area inside," interchanging area and volume ideas; others used the visual of filling a solid with cubes and counting the cubic units. Some referred to the previous object being 3D printed and discussed the number of layers being built up one at a time. Afterward, we engaged students in a unit-cube activity that allowed them to design any object they wanted in SketchUp. One limitation, however, was that they had to use only cubes of a given size. We showed them how to make a cube, then how to duplicate that cube as many times as was needed to make their object.

## Additional Lesson Ideas

The list below, although not exhaustive, shows some of the interesting ways that we believe 3D design and printing can be used to teach mathematical principles.

1. **CCSS.Math.Content.5.MD.C.3.BA:** *A solid figure, which can be packed without gaps or overlaps using  $n$  unit cubes, is said to have a volume of  $n$  cubic units.*

Assign students to work with a partner. Have students design a rectangular prism-shaped figure with dimensions  $3 \times 4 \times 3$  using unit cubes. Give one student 36 unit cubes and the other student 34 cubes. One cube will contain some gaps while maintaining the given dimensions. After printing the objects, ask the partners to compare the size and volume of their objects, encouraging them to notice that although two objects may have the same size or look the same outwardly, they may differ in volume (in this case, by 2 cubic units).

2. **CCSS.Math.Content.5.MD.C.5:** *Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.*

Ask students to watch as a 3D printer prints a rectangular prism, making sure they pay close attention to how the base is printed first (and the shape of the base), as well as how the printer builds the object's height as it adds layers to the base. As more plastic is added to the shape, the volume increases. Ask them how would they find the volume of a rectangular prism using multiplication? Using addition? How do they see this happening as the prism is printing?

3. **CCSS.Math.Content.5.MD.C.5.B:** *Apply the formulas  $V = l \times w \times h$  and  $V = B \times h$  for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.*

Direct students to see that the base is being printed  $h$  number of times, which is a multiplicative relationship. For a rectangular prism, the base is a rectangle with an area of  $l \times w$ ; we can also say that the volume of a rectangular prism may be found by multiplying  $l \times w \times h$ .

4. **CCSS.Math.Content.5.MD.C.5.C:** *Recognize volume as additive. Find volumes of solid figures composed of two nonoverlapping right rectangular prisms by adding the volumes of the nonoverlapping parts, applying this technique to solve real-world problems.*

a. Ask students to design a rectangular prism in SketchUp and write down its volume. Duplicate the prism several times to make an object out of multiple rectangular prisms. Find the volume of the new object. Why does this work?

b. Design and print two separate rectangular prisms with a volume of 24 cubic units. Design two rectangular prisms with the same volume of 24 cubic units, but this time, connect them to each other by stacking them on top of each other. Print all three objects, and compare the volume of the two separate prisms to the volume of the two connected prisms. Is it the same or different? Why?

5. **CCSS.Math.Content.6.RP.A.3:** *Use ratio and rate reasoning to solve real-world and mathematical problems.*

Discuss ratios and rates involved in the 3D printing process. Have students explore and compare the size of a 3D object, the speed at which it prints, the amount of filament used, and the price of the print.

6. **CCSS.Math.Content.8.G.A.1:** *Verify experimentally the properties of rotations, reflections, and translations.*

Have students each create 3D tessellation designs in SketchUp by performing rotations or translations with parts of a basic shape. If they design correctly, students should be able to 3D print multiple copies of their design and fit the tessellations together like puzzle pieces. Additionally, the design could be 3D printed and used as a stamp. If you used your stamp over and over, could you cover an entire page with no overlaps or white areas?

The students thought of the activity much like playing with Lego™ bricks. The duplicated cubes automatically “snapped” together like magnets when they were dropped close to each other. After the students had created their objects in SketchUp, we assigned

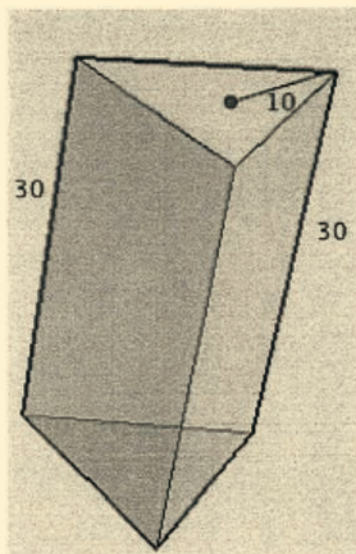
them to work with partners and gave the student groups a discussion guide to examine their creations in pairs. They were then asked to calculate the volume of their objects in cubic units, counting the number of cubes that they used.

Next, we took some time to examine a rectangular prism. We discussed how it was composed of cubic units as well. First, we asked students, “What is the shape of the base?” and referred to the 3D printing process by additionally asking, “What would



Fig. 3 Students analyzed their partner's design.

Design this triangular prism in SketchUp:



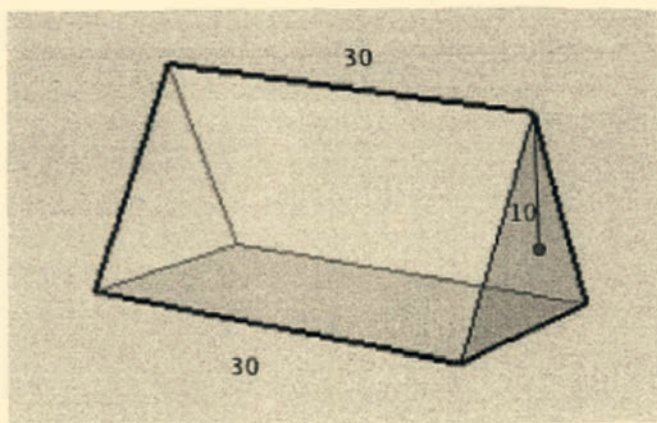
- Compare your shape with your partner's shape:

The measurements are the same but her's is on its side

(a)

### Cross-Sections

Design this triangular prism in SketchUp:



- Compare your shape with your partner's shape:

The measure is the same but it has different bases.

(b)

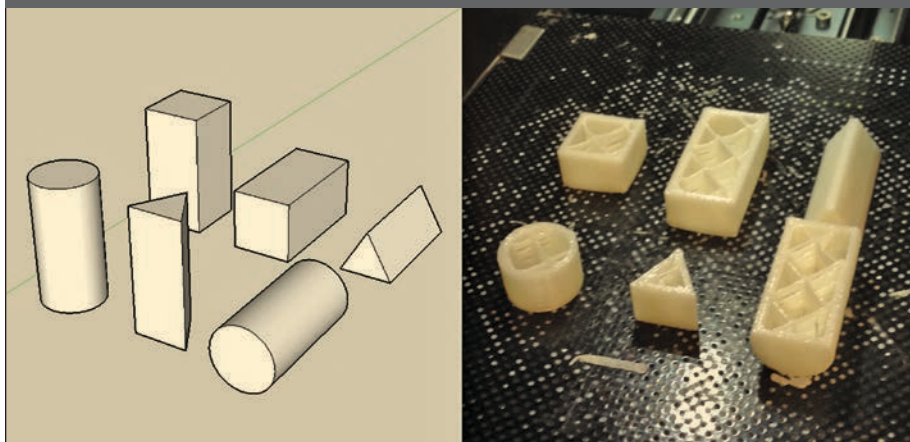
the first layer printed look like?"

We then showed the students video footage of a rectangular prism being 3D printed, encouraging them to pay attention to how the base, a rectangle, was printed first, followed by many layers that were added as the volume was increasing. We paused the video a few times to ask, "What shape is the printer printing in this layer now?" to encourage students to see that prisms comprise layers of the same polygon "stacked" on top of one another. One student, who was already making connections to the volume formula, commented on the algorithm, "You count length, width, height and multiply."

For our next activity, we placed the students in groups of four and asked each group to create either a triangular prism or a parallelogram prism in SketchUp. After students made their assigned prism, they were asked to consider the base of their solid and how they could find its area. Then, on the basis of what they already knew about volume and what they observed while the rectangular prism was printing, they were asked to develop their own strategy or formula to calculate the volume of their solid. They were also asked to decide which of their objects would take more plastic to print, connecting 3D printing once more to volume, and reinforcing the concept that increased volume is directly proportional to increased plastic (or whatever material is used to make the object).

The students understood that for a rectangular prism, the volume was equivalent to the area of the base multiplied by the height, so they hypothesized that for other prisms, they could also find the area of the base (a triangle, parallelogram, square, hexagon, pentagon, circle, and so on) and multiply that area by the height to find the prism's volume. We noticed that some students described it this way, but others continued to

**Fig. 4** Drawings next to the real objects made the 3D principle come alive.



say, “base times width times height” for each type of prism. As part of the discussion, we showed the students a time-lapse video of the 3D printer producing various prisms simultaneously, which helped them see that the volume of the prism began with the printer printing its base. As layers of plastic were added, students could see that the same base shape was being added repeatedly, eventually building to the height of the object. Seeing the 3D printer “creating” volume was extremely valuable in helping the students make these mathematical connections to the theoretical concepts of volume that they had been learning in the classroom. Every group of students eventually described the volume of a prism in terms of the area of the base multiplied by the height, specifically referencing the 3D printing of the base over and over.

After each group had developed a strategy or formula for finding their prism’s volume, we came back together as a whole class to let each group share its findings and receive feedback from the class. We tested and discussed the students’ different strategies, making connections to the 3D printing process as we went.

After the lesson, we received positive feedback from the fourth-grade classroom teacher, who described the lesson as a wonderful and exciting

experience for her students, in that “the way the lesson was structured really assisted the students in drawing conclusions. Most of them jumped right to the idea that you could take a base and, if you knew the area of that, you could then multiply height for volume.” Pleased with the lesson’s outcome, she expressed a desire to incorporate 3D printing into her curriculum in the future.

### CROSS-SECTIONS LESSON

A lesson implemented in the seventh grade classroom focused on geometric solids and their cross sections. The first objective was for students to define a cross section and describe the shape of a cross section in an applied context. The second objective was for students to determine the congruence of two geometric objects that have been designed in different orientations but still exhibit identical measurements.

The lesson began by associating the concept of cross sections to the 3D printing process as the students watched a time-lapse video of a 3D figure being printed. Because 3D printers build up an object layer by layer, students could visualize a cross section as one of those layers being printed at a specific moment. Next, pairs of students were each assigned a geometric solid of specific dimen-

sions. Students were instructed to use SketchUp to design their solid in a different orientation from their partner’s (see **fig. 3**).

After the design process, we asked students to compare their shapes with their partners’ shapes. One student noted, “The measurements are the same for both, but mine is on its side.” Another suggested, “They are the same shape, but my partner’s shape is up and down, not side to side.” After such discussions with their classmates, most students were able to hypothesize that their two solids were congruent, despite the different orientations. After the solids were printed, students could physically hold their design in their hands. We again asked students to compare their design with their partners’ design, so that they could verify that the shapes were congruent. This activity emphasized that a geometric solid is not necessarily restricted to the orientation represented on the page of a textbook or an activity sheet. With this dynamic approach provided by SketchUp, students could manipulate any solid in a variety of ways, such as stretching its size, changing its base, or viewing it from different angles. The classroom teacher found that this approach greatly facilitated the students’ abilities to grasp geometric concepts. She reflected, “For some students at this age, abstraction is still very much a struggle, so this provides depth and dimensions that they would not otherwise gain from a textbook.”

The printing process also reinforced the concept of cross sections. Before printing, we asked students what the cross sections of their shapes would look like when cutting parallel and perpendicular to the shape’s base. Then we printed six geometric solids at once—three different solids, each oriented two different ways (see **fig. 4**). As the objects printed, we asked students to conjecture which solid they had



designed and which solid their partner had designed, based on the shapes of the layers being extruded by the 3D printer. For example, students who had designed the upright cylinder immediately recognized their shape being printed because it was the only shape with circular cross sections parallel to the print platform. However, it took longer for students to distinguish the rectangular prism and the triangular prism with rectangular layers being extruded by the printer because at first the cross sections were very similar.

By making these mathematical observations, students could compare the cross sections of their solid with its congruent counterpart and also with the cross sections of other geometric solids. Such discussions led to conversations about which orientation was most practical when 3D printing a particular solid.

Because of the students' prior difficulties in understanding cross sections, the teacher recommended that we teach a lesson that would help build their spatial reasoning. This lack of understanding was confirmed when the students struggled to describe cross sections at the beginning of this lesson.

After the students were provided with 3D printing as a physical context in which to think about cross sections, they were more comfortable visualizing and describing cross sections of various objects, often discussing how the object would look as the 3D printer created each layer. Additionally, the lesson allowed students to compare different cross sections of multiple geometric solids first by design, then by observing the 3D printing process, and finally by examining the physical printed product. Each step furthered the students' notions of cross sections and the connections between various geometric solids.

## GEOMETRIC PRINCIPLES REINFORCED THROUGH 3D PRINTING

We continue to learn more about 3D printing and its connections to geometric content. We have found three significant properties of 3D printing that can be used in future lessons with a variety of topics. The first of these properties—additive, layered construction—relates to such geometric ideas as volume, height, cross -sections, and surface area. A second property is that 3D printing is a process. Thus, it can be paused, measured, and optimized by taking into account such variables as time, amount of plastic, size, cost, and so on. The third property of 3D printing that arose from our work relates to the software that we use to design objects. There is a meaningful use of measurement tools and concepts such as rulers, protractors, and the differences and connections between two- and three-dimensional objects including length, area, and volume measurements. For these reasons, we are excited about this technology's potential to inspire and challenge students to learn geometry concepts in a new and creative way. (See the **sidebar** on p. 539 for Additional Lesson Ideas.)

### CCSSM Practices in Action

CCSS-M.5.MD.C5  
CCSS-M.7.G.A

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