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Technology-Based Geometry Activities for Teaching Vector Operations

Although linear algebra has been included in the high school curriculum (CCSSI 2010), better support is needed for teachers. Most textbooks are written for college students and emphasize heavy computations, algorithms, and procedures. A newspaper article (Mathews 2012) described a seasoned high school teacher struggling with teaching linear algebra, making mistakes, being confused, and eventually quitting his job midsemester. The school administration scrambled to find a replacement, going from substitute teachers to mathematics graduate students (from a nearby university) to its own mathematics chair, who had no previous experience teaching linear algebra either.

To address the need for teacher support and effective learning experiences for students, we present four activities that are specifically focused on making linear algebra intuitive, engaging, and hands-on. The topics are included in the vector and matrix quantities strand of the Common Core State Standards for Mathematics (CCSSI 2010):

- Recognize vector quantities as having magnitude and direction (CCSSI.MATH.HSN.VM.A.1)
- Addition and subtraction of vectors (CCSSI.MATH.HSN.VM.B.4)
- Scalar multiplication (CCSSI.MATH.HSN.VM.B.5)

We used The Geometer's Sketchpad® (GSP), although GeoGebra could also be used. Students working on these activities will need interactive geometry software skills for constructions including measurements, parameters, and calculations. We encourage teachers to implement all four activities, which took about seventy minutes, in this order, particularly because students working on them have demonstrated improved learning and understanding of vector operations (see Appova and Berezovski 2013). Discussion questions are included to encourage students to reflect on their observations.

ACTIVITY 1: SCALAR MULTIPLICATION

Defined as a directed line segment, a vector with its initial point (or tail) at the origin can be represented by the coordinates of the segment's endpoint (or tip). Thus, calculations in this activity used point coordinates to represent a vector, $\vec{v}$. Students manipulated a parameter, c, to create a dynamic segment, $c\vec{v}$. With attention to the cases $c > 1$, $c < 0$, and $0 < c < 1$, students recorded the scalar values c and noted the corresponding effect on the vector $c\vec{v}$.

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Activities for Students appears five times each year in *Mathematics Teacher*, often providing reproducible activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information, visit http://www.nctm.org/mtcalls.

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The purpose was for the students to make explicit connections between the geometric representation of vectors and multiplication by a scalar as they noticed expansions, compressions, and changes in orientation. (See **fig. 1**.)

Discussion Questions

- What scalar c will make the directions of the vectors $\vec{v}$ and $c\vec{v}$ different? Explain why this occurs.
- What scalar c will make the sizes of the vectors $\vec{v}$ and $c\vec{v}$ different but will keep their directions the same? Explain.
- On the basis of your observations, identify the vectors $c\vec{v}$ with the same size as $\vec{v}$.
- Describe all the vectors $c\vec{v}$ with the same direction as $\vec{v}$.

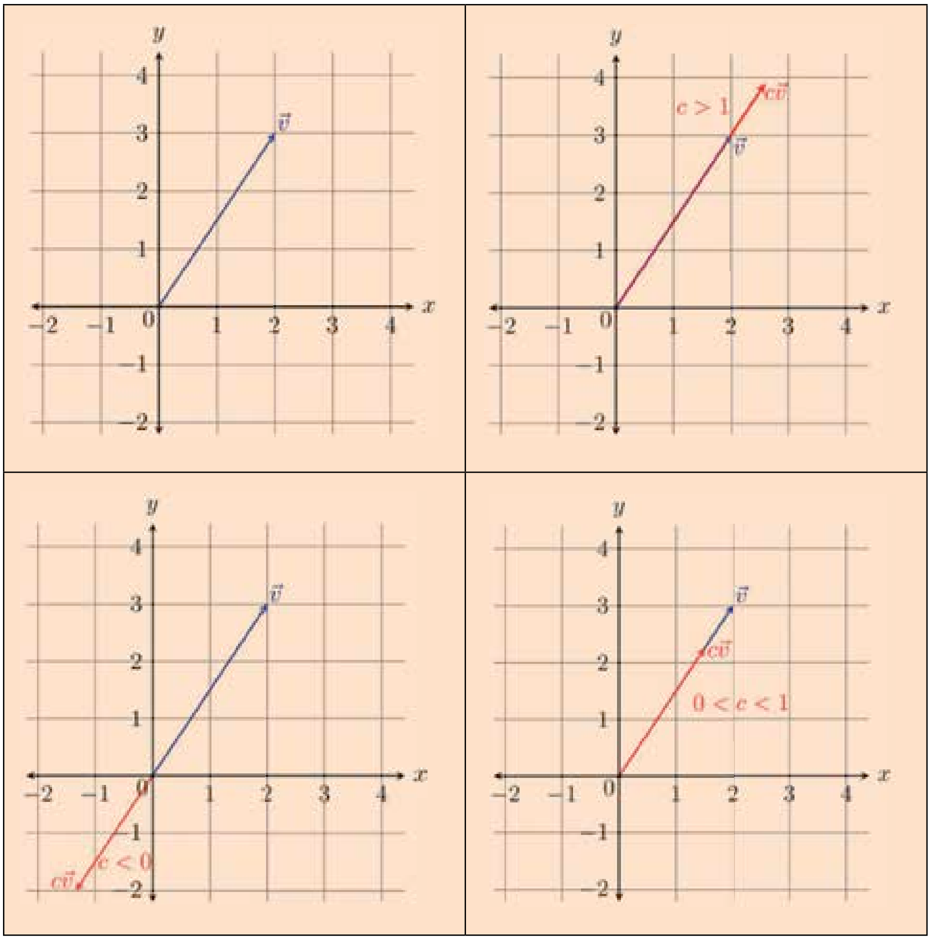

Fig. 1 Samples of screen shots from the scalar multiplication activity show the effect of different values of *c*.

ACTIVITY 2: VECTORS IN DILATIONS AND CONTRACTIONS OF SHAPES

This transitional activity helped strengthen students' ideas about scalar multiplication as they generalized from line segments to geometric shapes. By doubling and halving the vertex coordinates of a given rectangle, students directly observed and explained the meaning of dilation and contraction as the stretching and shrinking of geometric shapes.

To begin, students constructed rectangle $A_1B_1C_1D_1$ with coordinates that are obtained from the coordinates of $ABCD$ by scalar multiplication (see **fig. 2**). Students examined the sizes of the rectangles and considered the proportional changes and scale factors of the sides. Stretching or shrinking the original (blue) rectangle resulted in the similar scaled (green and red) objects.

Discussion Questions

- Construct a rectangle $A_1B_1C_1D_1$ by doubling the coordinates of rectangle $ABCD$. Measure the lengths of the sides of the original rectangle and the new rectangle. What do you notice?
- Explain the relationship between the corresponding vertices of rectangles $ABCD$ and $A_1B_1C_1D_1$ in terms of vectors.
- Construct the rectangle $A_2B_2C_2D_2$ by halving the coordinates of $ABCD$. How are the three rectangles that you have created related? How does the relationship among these rectangles relate to vectors?
- In your own words, explain the meaning of a *dilation* and a *contraction*.

ACTIVITY 3: VECTOR ADDITION

This activity targeted students' understanding of vector addition through geometric constructions, manipulations, and observations using technology. The software allowed students to move the vectors $\vec{u}$, $\vec{v}$, $\vec{v}+\vec{u}$, and $\vec{u}+\vec{v}$ around the screen while tracking the coordinates (see **fig. 3**). Using technology, students

Fig. 2 A screen shot shows dilations and contractions of the blue rectangle.

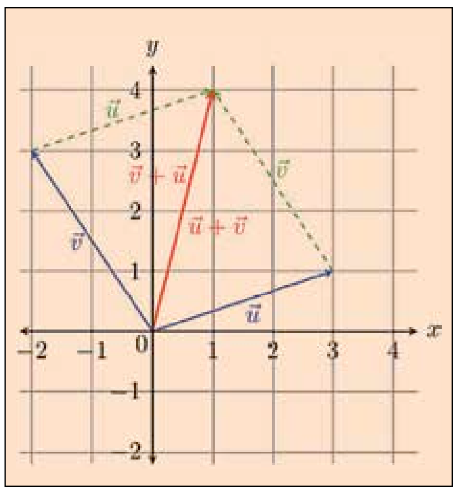

Fig. 3 The vector addition activity explores commutativity using the triangle and parallelogram constructions.

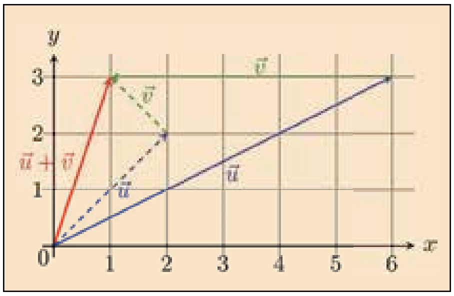

Fig. 4 The resultant $\vec{u}+\vec{v}$ (in red) may be longer than both $\vec{u}$ and $\vec{v}$ (shown as dashed segments) or shorter (shown with solid segments).

explored and answered construction-based questions.

From class, many students were aware of the coordinate addition rule. However, they also needed to be able to use the software to geometrically construct the resultant vector $\vec{v}+\vec{u}$ or $\vec{u}+\vec{v}$, given vectors $\vec{u}$ and $\vec{v}$. Geometric interpretations (i.e., the triangle and parallelogram rules) helped students observe and explain why vector addition is commutative. Some students identified the parallelogram as being composed of two congruent triangles; some noted that opposite sides of a parallelogram are congruent. Discussion led students to recognize that the decomposition of a vector into a sum of two vectors is not unique. (See **fig. 4**.)

As part of this activity, teachers might ask students to complete the following tasks:

- Using the coordinate addition rule, add vectors $\vec{u}$ and $\vec{v}$ to construct the resultant vector $\vec{u}+\vec{v}$.
- Construct the vectors $\vec{v}+\vec{u}$ (by adding $\vec{u}$ to the tip of $\vec{v}$) and $\vec{u}+\vec{v}$ (by adding $\vec{v}$ to the tip of $\vec{u}$). Compare these vectors, $\vec{v}+\vec{u}$ and $\vec{u}+\vec{v}$, with the vector that you constructed using the coordinate addition rule.

Discussion Questions

- Explain why the different constructions adding $\vec{u}$ and $\vec{v}$ result in the same vector.
- Change the magnitudes of $\vec{u}$ and $\vec{v}$ to check that commutativity still holds. In your own words, explain why addition of vectors is commutative.
- Is it possible to find other pairs of vectors (different from the given $\vec{u}$ and $\vec{v}$) that will add up to the same resultant? If so, use technology to construct an example. If not, explain why not.
- Is it possible to find a pair of vectors $\vec{u}$ and $\vec{v}$ so that $\vec{u}+\vec{v}$ is shorter than $\vec{u}$ and shorter than $\vec{v}$? If so, use technology to construct an example. If not, explain why not.
- Is it possible to find a pair of vectors $\vec{u}$ and $\vec{v}$ so that $\vec{u}+\vec{v}$ is longer than $\vec{u}$ and longer than $\vec{v}$? If so, use technology to construct an example. If not, explain why not.

ACTIVITY 4: VECTOR SUBTRACTION

In this activity, students applied knowledge from the previous activities: scalar multiplication, construction of opposite vectors, and vector addition. They considered vector magnitude, direction, and coordinates. Students made specific connections between a geometric representation of the triangle or parallelogram rule and the addition or subtraction of vectors (see **fig. 5**). Some students noted that subtracting a vector is the same as adding its opposite. In these explorations, the software allowed students to move the vectors but keep their magnitude and direction fixed (unchanged). Students could be asked to complete the following tasks:

- Construct a vector $\vec{w}$ starting at the terminal point of $\vec{v}$ and connecting to the terminal point of $\vec{u}$. Provide its coordinates and magnitude.
- Construct a vector $\vec{s}$, which is the sum of $\vec{u}$ and $-\vec{v}$. Provide its coordinates and magnitude.

Discussion Questions

- How do $\vec{u}$, $\vec{v}$, and $\vec{w}$ relate in terms of their coordinates and magnitudes?
- How do $\vec{w}$ and $\vec{s}$ relate in terms of their coordinates and magnitudes?
- Given another set of two vectors $\vec{u}$ and $\vec{v}$, explain how you would create a vector $\vec{w}$ such that $\vec{w}=\vec{u}-\vec{v}$.

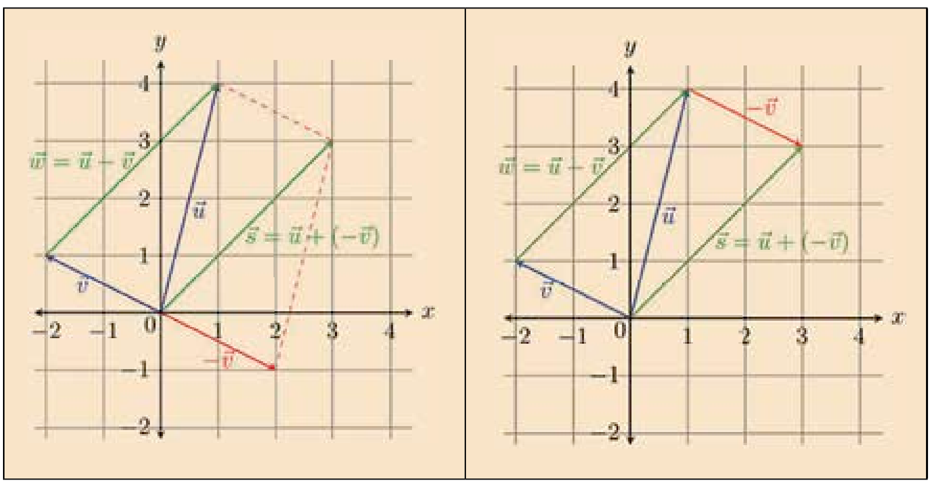

Fig. 5 Sample screen shots show vector subtraction.

GEOMETRIC INTERPRETATION IN LINEAR ALGEBRA

Research studies strongly emphasize the use of geometric structures and representations in linear algebra (Harel 1989; Tabaghi 2010; Gueudet-Chartier 2002). These four activities target students' learning through geometric interpretations of vectors (Tabaghi 2010), the development of students' concept images (Harel 1989), and geometric intuitions (Gueudet-Chartier 2002). The discussion questions are included to help teachers situate linear algebra in a more engaged and hands-on learning environment in their classrooms. As an additional resource, a Geometer's Sketchpad file is available with the online article.

REFERENCES

Appova, Aina, and Tetyana Berezovski. 2013. "Commonly Identified Students' Misconceptions about Vectors and Vector Operations." *In Proceedings of the Thirty-Second Annual Conference of the SIGMAA on Research in Undergraduate*

Mathematics Education (*RUME*), vol. 2, pp. 2–8.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Conley, David T. 2007. *Redefining College Readiness*. Eugene, OR: Educational Policy Improvement Center.

Gueudet-Chartier, Ghislaine. 2002. "Using 'Geometrical Intuition' to Learn Linear Algebra." *Proceedings of CERME,* vol. 2, pp. 533–41. Prague: Charles University.

Harel, Guershon. 1989. "Learning and Teaching Linear Algebra: Difficulties and an Alternative Approach to Visualizing Concepts and Processes." *Focus on Learning Problems in Mathematics* 11: 139–48.

Mathews, Jay. May 5, 2012. "Math Stumble at Renowned Jefferson High." *Washington Post*. https://www.washingtonpost.com/blogs/class-struggle/post/math-stumble-at-renowned-jefferson-high/2012/05/05/gIQATYUQ4T_blog.html

Tabaghi, Shiva Gol. 2010. "The Use of DGS to Support Students' Concept Image Formation of Linear Transformation." In *Proceedings of the Thirty-Second Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education* (*PME-NA* 32), vol. 6, pp. 1318–26.

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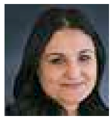
For a Geometer's Sketchpad file, go to the online article at **www.nctm.org/mt**.

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