STAIRCASES, TOWERS, AND CASTLES

Geometric patterns are generalized recursively and explicitly to relate multiple representations modeled by quadratic expressions.

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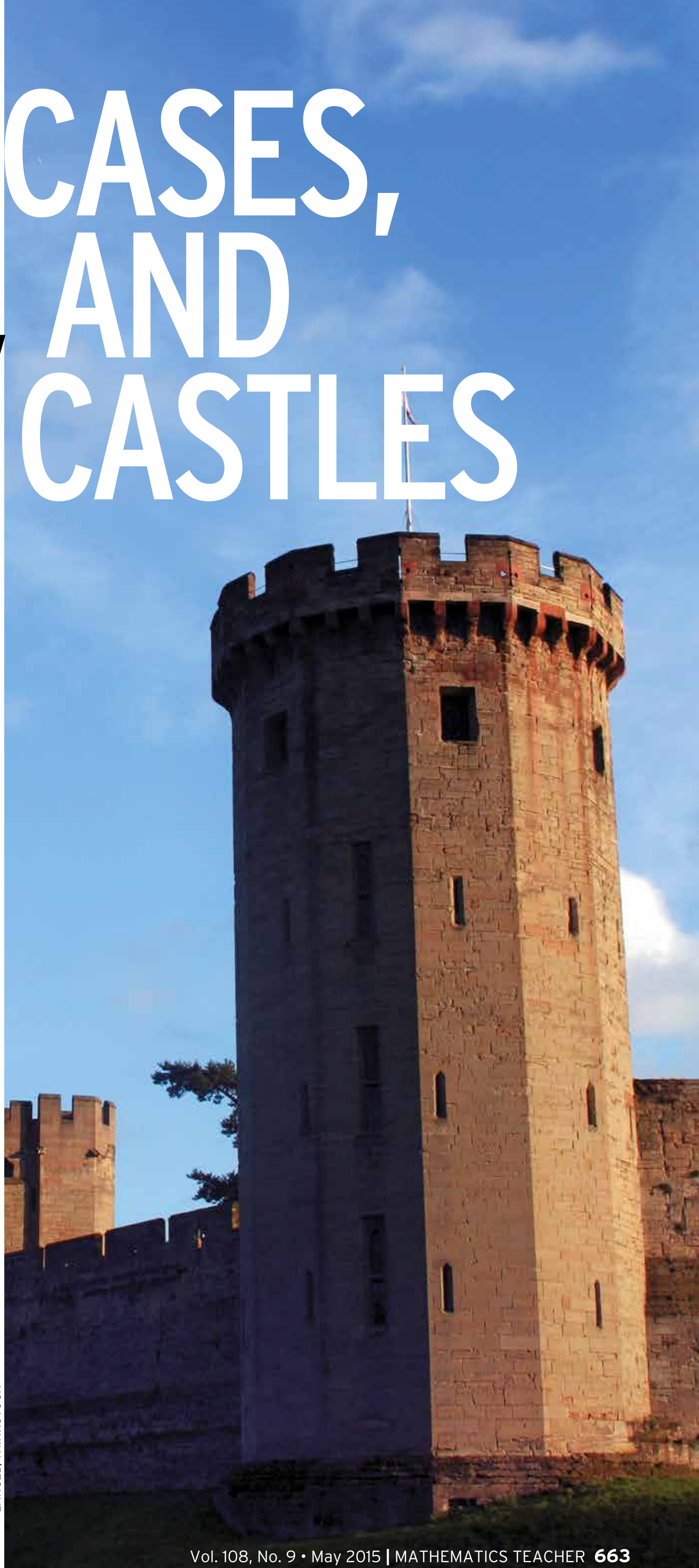

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The very nature of algebra concerns the generalization of patterns (Lee 1996). Patterning activities that are geometric in nature can serve as powerful contexts that engage students in algebraic thinking and visually support them in constructing a variety of generalizations and justifications (e.g., Healy and Hoyles 1999; Lannin 2005). In this article, we discuss geometric patterning tasks that engage students with wide-ranging levels of ability, interest, and motivation. This succession of tasks is likely to elicit recursive reasoning strategies to build mathematical sequences on previous terms' values or explicit formulas to determine any value in the sequence. The tasks are increasingly complex in terms of mathematical patterns, numeric computations, and visualization demands.

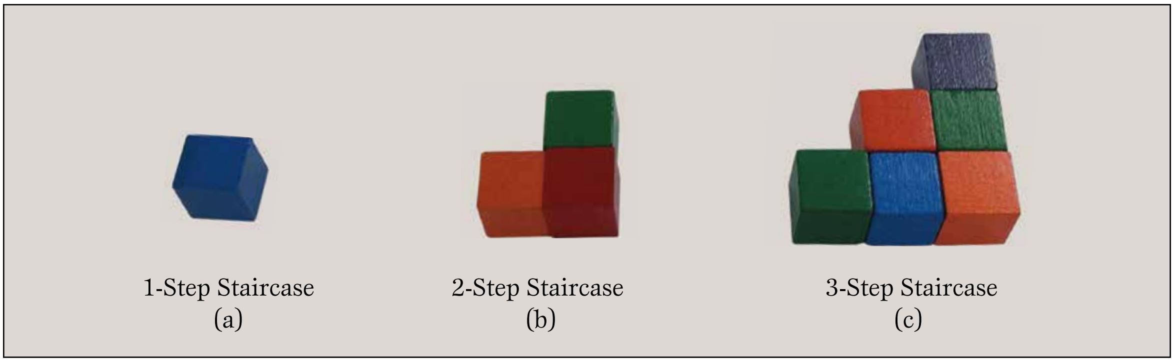

Fig. 1 Cubes are used to build the first three staircases.

In this article, we authors—three researchers and a classroom teacher—share how we structured and reflected on our implementation of this task sequence in an Algebra 1 class. We organized students into groups of three and gave each student a role: A recorder would keep track of the group's ideas; a builder would build and help explain the structures; and a reporter would share the group's ideas. We made cubes, graph paper, and calculators available. Each group was given opportunities to present its ideas to the class at various points during the solution process.

For the Algebra 1 class discussed here, we presented two of the tasks over three class periods.

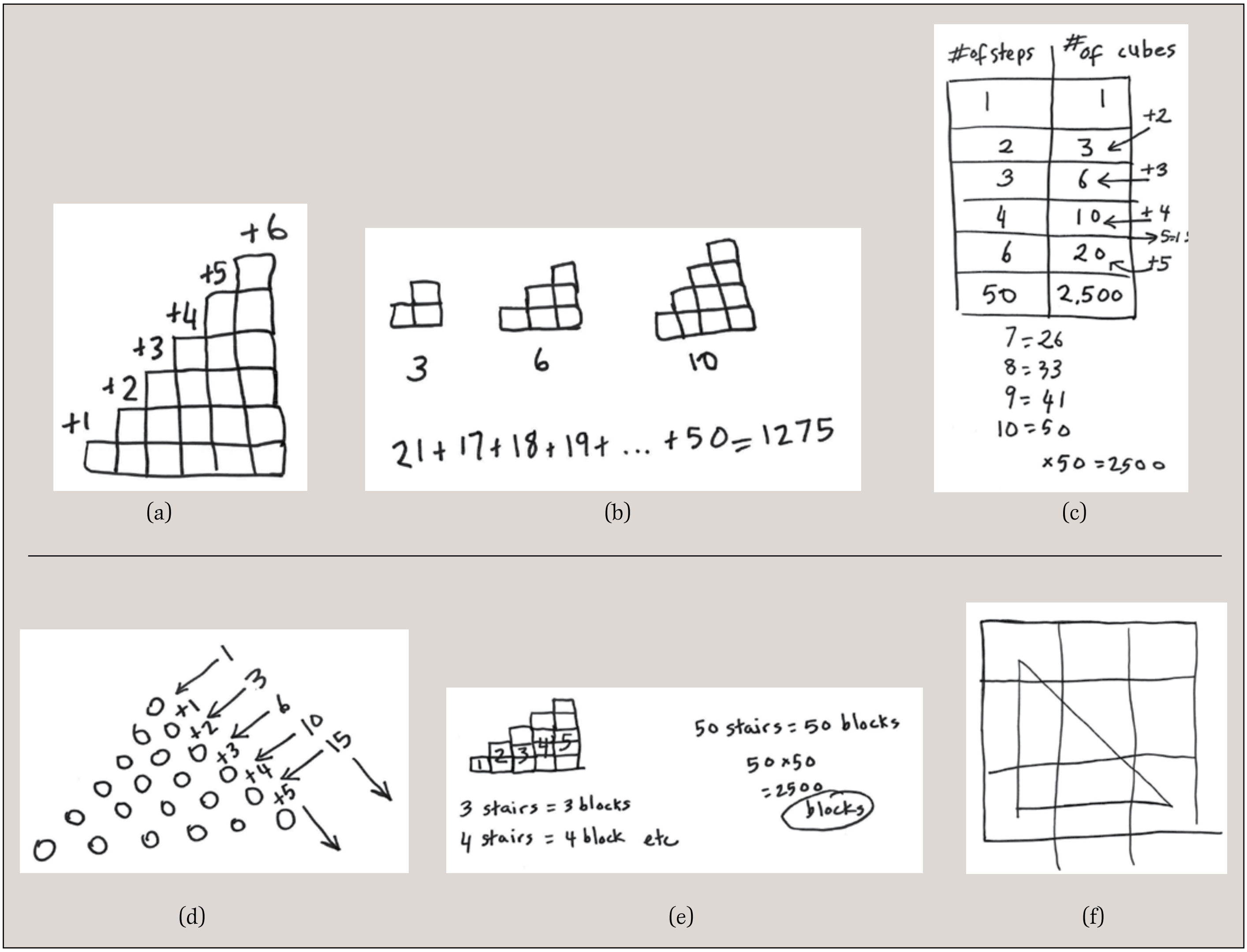

Fig. 2 Students begin solutions to the Staircase task.

DAY 1

Task 1: Staircases

A group of students are building staircases out of wooden cubes. The 1-step staircase consists of one cube, and the 2-step staircase consists of three cubes stacked (see **fig. 1**). How many cubes will be needed to build a 3-step staircase? A 6-step staircase? A 50-step staircase? An n-step staircase?

While working on the task, students needed to visualize or represent the isosceles right-triangle-shaped staircases by building them with cubes or by drawing them. After approximately twenty minutes of exploration time and before any of the groups had generated an explicit expression for the number of cubes needed to build an n-step staircase, we invited each group to share its partial solutions. Five of the six groups shared their ideas by drawing and writing on the board. Three of the groups produced a numerical answer for the number of cubes needed to build the 50-step staircase. Two groups came up with an answer of 2500, and one group came up with the correct answer of 1275. (See **figs. 2a–e** for written records of the solutions from five groups.)

Figure 2a represents a group's drawing of the 6-step staircase and the initial phase of what these students noticed in the pattern. Another group attempted to add the number of cubes by using a recursive strategy of summing consecutive integers (see **fig. 2b**). Although this group obtained the correct numeric answer of 1275 for the 50-step staircase, its records had an error ($21 + 17 + 18 + 19 + \cdots + 50 = 1275$ instead of $21 + 7 + 8 + 9 + \cdots + 50 = 1275$). **Figure 2c** is an example of another group's numerical representation of a recursive strategy. Students in this group recorded the number of cubes for each staircase as well as the number of cubes added for each subsequent step. However, they reported the 6-step staircase incorrectly. Later, they multiplied 50 times 50 and concluded that 2500 cubes were needed for the 50-step staircase, an incorrect answer. Another group determined the number of cubes for the first five staircases by drawing the diagram in **figure 2d**. The last group of students thought about the 50-step staircase as a $50 \times 50 \times 1$ wall; they seemed to be thinking about a square number of 50×50 (see **fig. 2e**).

One student, Brian (all students' names are pseudonyms), noticed that the 3-step staircase could be transformed into a square with side length of 3 by combining the 2-step and 3-step staircases, but he was not able to use what he noticed to make a generalization about the pattern (see **fig. 2f**).

This task was a challenge for students. Although they quickly and easily built the 1- through 6-step staircases and drew records to keep track of their ideas, most students struggled to determine the number of cubes that would be needed to build the 50-step staircase. Many students focused on looking for a numerical pattern rather than making use of the geometric shape of the staircases. The classroom teacher's reflection about day 1 reveals the challenge of supporting secondary school Algebra 1 students in generating an explicit formula.

> Overall, students understood that you can determine the number of blocks by adding $1 + 2 + 3 + \cdots + n$ number of steps, but they didn't know what to do to go beyond that recursive formula. I'm worried that it will be difficult to push them toward an explicit formula. Students were asking whether I would tell them the "answer" at the end of today's lesson. . . .

Although students struggled with writing an explicit expression, four of the six groups made significant advances toward a viable solution. After reflecting on the groups' partial solutions, we decided to begin the next class by facilitating a class discussion about two of them—one from a group that relied on recursive thinking (see **fig. 2b**) and another from a group that relied on reasoning about the underlying geometric structure (see **fig. 2f**), which we thought could support an explicit way of thinking about the pattern.

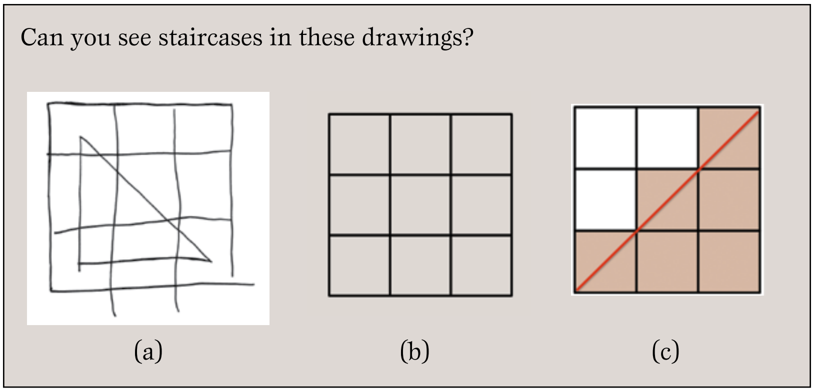

Fig. 3 A shaded diagram related to Brian's work helps clarify his idea.

DAY 2

At the beginning of the next class, we reviewed the partial solutions that had been shared the previous day and asked students to discuss the other groups' ideas. All the groups quickly agreed that the number of cubes needed to build the 50-step staircase could be found by adding all the consecutive integers from 1 to 50. At this point, we returned to Brian's drawing (see **fig. 2f**) to encourage a class discussion aimed at generating an explicitly defined formula and to link this formula to the geometric representation of the pattern.

We created some additional drawings (see **fig. 3**) to help students make sense of Brian's way of thinking about composing two different staircases to form a square. However, the students were not yet able to use what they noticed to make a generalization about the pattern. We next suggested a strategy of composing two of the same staircases to form a rectangle. **Figure 4** illustrates a rotated and translated 3-step staircase on the top of another 3-step staircase, part of the animation that we presented.

We asked students to guess how many cubes there are in one 3-step staircase. Students immediately recognized that for two 3-step staircases, there should be 3 times 4 cubes; for one 3-step staircase, that number should be divided into 2. They later applied this geometric approach to larger staircases and talked about whether they could generate a formula based on this approach. Some students came up with the solution for two n-step staircases (by imagining rotating and translating two n-step staircases) to form an $n \times (n + 1)$ wall, providing justification for the explicit expression $n(n + 1)/2$.

For a recursive solution, we introduced the method of adding

$$1 + 2 + 3 + 4 + \cdots + 10 = A$$

and

$$10 + 9 + 8 + 7 + \cdots + 1 = A,$$

resulting in ten 11s for the sum of two As. For one A, this sum should be divided into 2. This method can be generalized to the n-step staircase to produce n pairs, each of which has a sum $(n + 1)$. This method was challenging for most of the students. However, they were involved in the discussion and felt comfortable sharing their thoughts for each step. The questioning style was crucial for each step. Rather than just stating the answer, we wanted them to think about the methods and their generalizability. The reflection of the classroom teacher indicated how this instruction helped some students in the classroom:

> When asked the meaning of dividing by 2, a few students indicated that it was because we were multiplying. I decided to point out that it was related to the picture, and one student stated, "There's two staircases." That student seems to be engaging really well with this activity. She may be a prime example of a student who struggles with the traditional model of the classroom, but she was excelling at this problem-solving activity.
>
> The problem-solving model appeals to many different types of learners. There is the visual representation of pictures, the kinesthetic representation of the blocks, and the auditory representation when the groups share. . . . The atmosphere of the

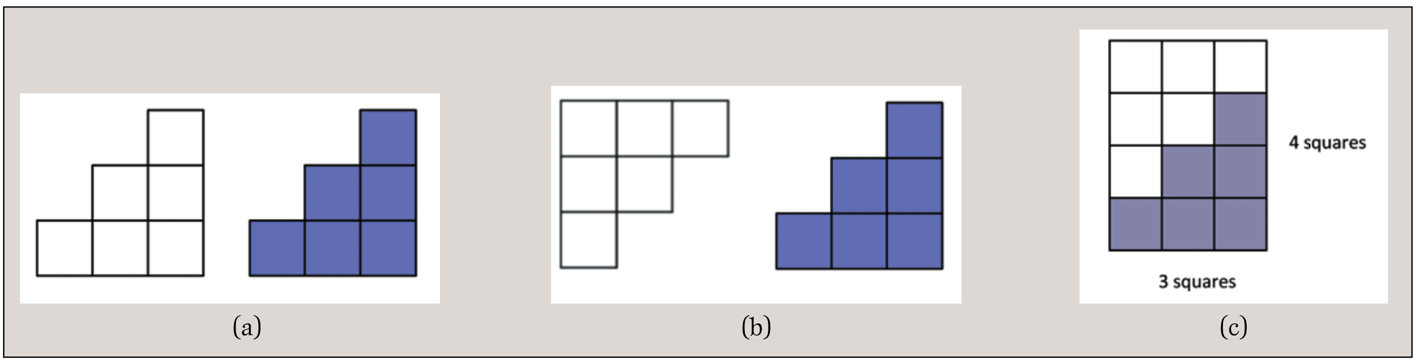

Fig. 4 Snapshots of an animation suggest a method using two identical staircases (**a**), a rotation (**b**), and a translation (**c**).

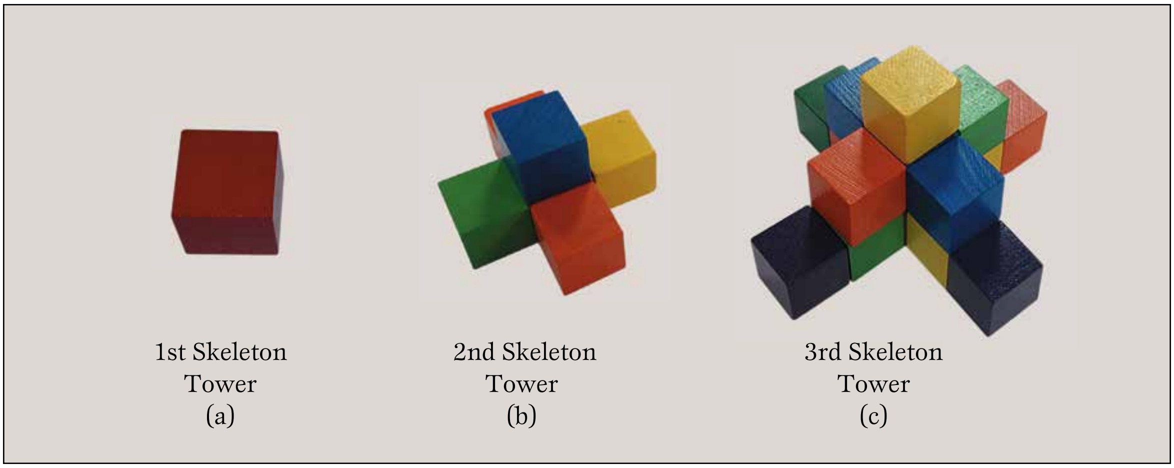

Fig. 5 Skeleton towers extend from a central stack.

> classroom at the end of day 2 had changed significantly since the end of day 1. We can only see what will happen in day 3.

After the students generalized the staircase pattern in both recursive and explicit ways in the first part of day 2, we posed the Skeleton Towers task.

Task 2: Skeleton Towers

> A skeleton tower is made up of a stack of cubes with a triangular wing on each of the four lateral faces of the cube. The pictures represent the first three skeleton towers (see **fig. 5**). How many cubes would be needed to build the 4th skeleton tower? The 5th? The 10th? The *n*th? How would you describe the pattern verbally?

As with the Staircase task, many students needed to represent the skeleton towers by building them with cubes or by drawing them. We encouraged students to imagine the towers from a bird's-eye view and keep track of the number of cubes in each "stack" using the convention displayed in **figure 6**, introducing this as a method that the recorder in another class used to represent some of her group's thinking. At the end of day 2, students were still working on the tasks.

DAY 3

At the beginning of day 3, we asked groups to work again on the Skeleton Towers task and later share their ideas about the tasks. **Figure 7a** represents the solution of one group for the 5th skeleton tower. Two groups generated expressions for the number of cubes in the *n*th skeleton tower by thinking about the staircases as the parts of the tower (see **figs. 7b** and **7c**). They then used what they remembered about the staircase pattern to help them describe this new pattern.

In general, the number of cubes needed to build the *n*th skeleton tower is 4 times the number of cubes in the $(n - 1)$-step staircase plus n, or

$$4\left(n-1\right)n/2+n=2\left(n^2-n\right)+n=2n^2-n,$$

for positive integers, n.

The excerpt below from the teacher's reflection on day 3 reveals the change in the classroom atmosphere as well as classroom norms.

> After finishing the Staircase activity from day 2, the students "knew" how to approach the Skeleton Towers activity. There was great enthusiasm in this environment, and it was apparent that students understood the roles and the norms of the classroom. Although students were willing to share and discuss, there were still students who did not want to present "wrong" information. This may have been instilled within my classroom as well as their previous math experiences. Students may view math problems as having only one answer. In problem solving, there is not one answer or one approach to a situation. It is necessary to create an environment within the classroom where students are not afraid to try things. We have seen over the past three days that even the smallest drawing on the back of a sheet

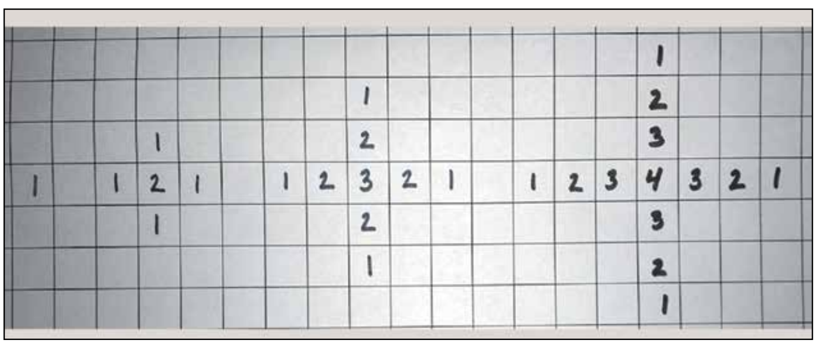

Fig. 6 A bird's-eye-view report allows students to record the number of cubes.

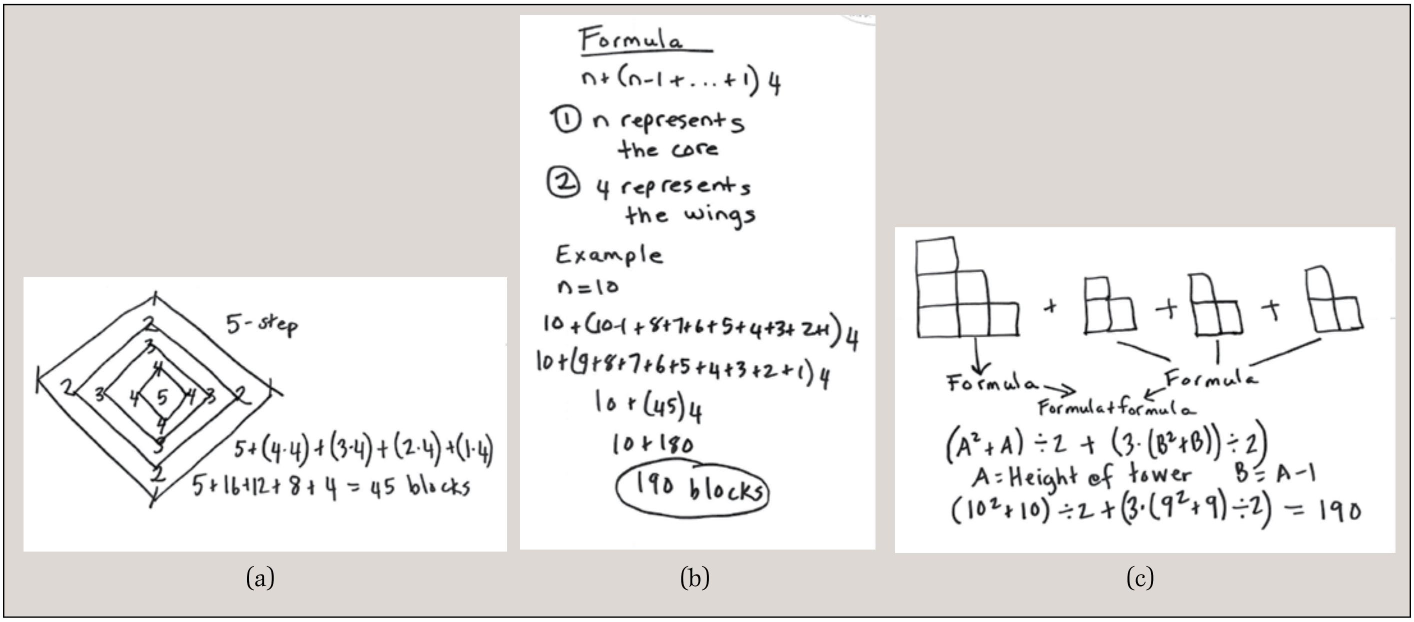

Fig. 7 Students solve the Skeleton Towers task.

of paper may signify deep processing within the mathematics.

Depending on the challenge level that students can handle, they can be introduced to a third task in the sequence. We did not introduce the third task because of time constraints.

Task 3: Skeleton Castles

A skeleton castle is made up of stacks of cubes that rise on four corners of a square and descend to meet at the middle of each side of the base. Skeleton castles that are 2 cubes high, 3 cubes high, and 4 cubes high are shown (see **fig. 8**). How many cubes would be needed to build a skeleton castle that is 5 cubes high? 6 cubes high? 10 cubes high? n cubes high?

One way to solve this task is to decompose the castle into four stacks of height n, one at each corner of the square base, and eight staircases of height $(n-1)$, two on each of the four sides of the square base. Because each staircase shares the first step, which is a single cube, the first step is double-counted four times. In total, the number of cubes needed to build the n-high skeleton castle is

$$4n + 8(n-1)n/2 - 4 = 4n^2 - 4,$$

for integers $n > 1$.

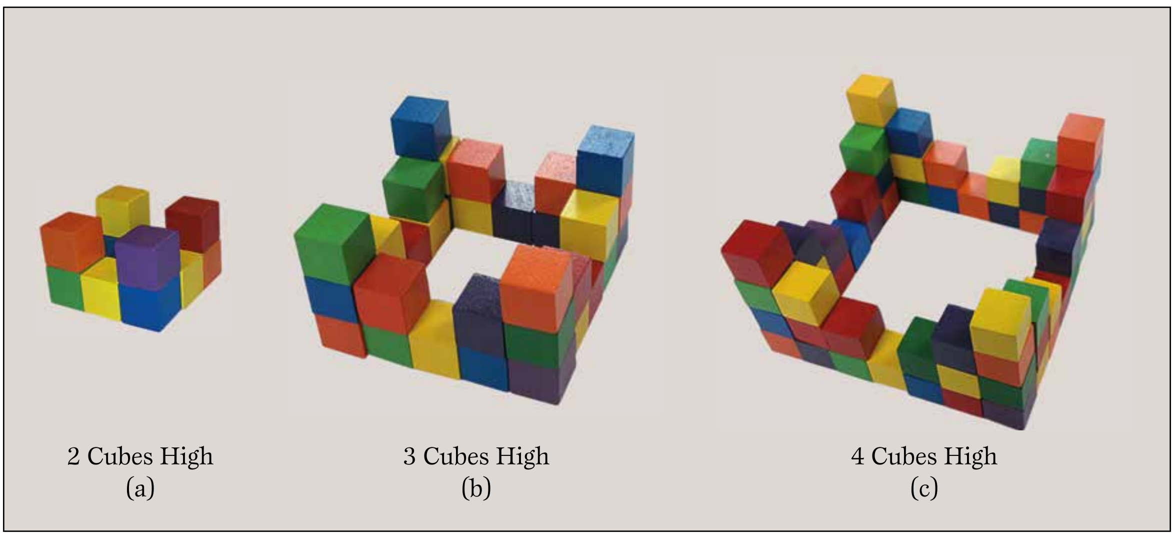

Fig. 8 Skeleton castles have a square footprint.

TASK EXTENSIONS

The tasks described here encourage students to reason algebraically about geometric structures in three-dimensional space. However, each task can be extended to engage students in reasoning about measurement, specifically length and area. The extension prompts include this problem:

> Use your drawing of a bird's-eye view to help you think about the footprint of a staircase (or skeleton tower or castle). What is the area (or perimeter) of the footprint of a 1-step staircase? A 2-step staircase? A 50-step staircase? An n-step staircase?

Further task extensions could be couched in real-world contexts involving predicting the height of a growing plant (or fantasy creature), determining the number of rooms in buildings with varying numbers of floors, or constructing borders of gardens of varying shapes and areas. Other scenarios could extend beyond quadratic relationships to include modeling the growth of populations or investments. For example, students could examine past data concerning the number of smartphones sold each year to predict future sales. In each situation, cubes could provide visual support as students quantify the patterns by making sense of geometric structures.

THE LESSON'S PEDAGOGICAL BENEFITS

The teacher's reflections and the students' solutions illustrate the instructional value of this three-day lesson by showing that students gained confidence and sophistication in generalizing and justifying patterns. Day 1 was the most difficult day; however, as the teacher noted on day 2, the classroom atmosphere "changed significantly since the end of day 1" as students successfully generalized the Staircases problem pattern in a variety of ways. By day 3, the teacher noted that students "knew how to approach" the Skeleton Towers problem with strategies they refined earlier.

The instructional value of this lesson is further substantiated by its relevance to the Common Core State Standards for Mathematics (CCSSM) treatment of functions (HSF.BF.A.1 and HSF.BF.A.2) and the five CCSSM Standards for Mathematical Practice (SMPs) (CCSSI 2010, pp. 6–8). To support students in making sense of problems and persevering in solving them (SMP 1), we withheld strategy hints and answers. Instead, we allowed time for small-group and whole-class discussions to encourage students to construct viable arguments and critique the reasoning of others (SMP 3). We supported students as they modeled with mathematics (SMP 4) while looking for and making use of structure (SMP 7) by providing different ways to represent the structures (such as building cubes and the bird's-eye-view method of reporting). We found that, for each task, building the pattern helped students reason abstractly and quantitatively (SMP 2). This unique and carefully sequenced set of tasks allowed students to relate numeric (sequences and series), algebraic (explicit expression), and geometric (a building or drawing) structures.

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