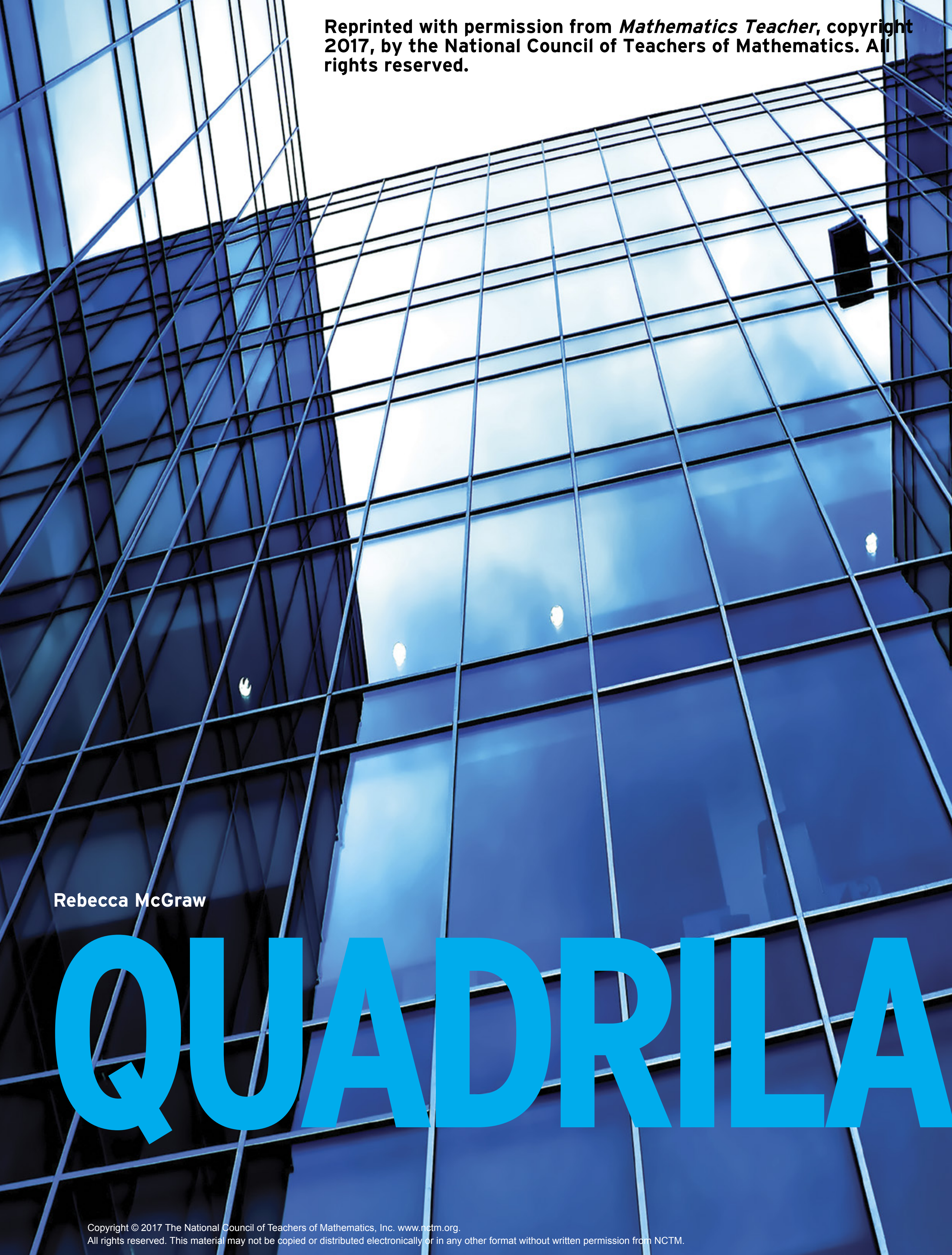

Rebecca McGraw

QUADRILA

Students investigate geometric relationships using such strategies as dissecting-and-rearranging.

The task I share here provides geometry students with opportunities to recall and use basic geometry vocabulary, extend their knowledge of area relationships, and create area formulas. It is characterized by reasoning and sense making (NCTM 2009) and the "Construct viable arguments and critique the reasoning of others" Mathematical Practice from CCSSI (2010).

Geometry students should engage in rich mathematical tasks that help them develop sense-making and problem-solving skills while building knowledge of geometry concepts and vocabulary (*Principles and Standards for School Mathematics* [NCTM 2000]; Common Core State Standards for Mathematics [CCSSI 2010]). Characteristics of rich tasks include multiple entry points, connections across representations and mathematical concepts, multiple solution paths, and interesting and perhaps surprising results that illuminate important mathematical ideas.

THE DIAGONAL AND AREA TASK

The following description of how people in ancient civilizations determined the areas of a plot of land offers one way to introduce the task by making a connection to real life. (The mathematical method described can be found in Crawford [1971]; see also Robson [2008].)

> Determining area has been a real-life problem for humankind since ancient times. Finding the area of a field is important for planting crops, estimating harvests, and determining taxes. A method used by the ancient Egyptians and Mesopotamians to estimate the area of a four-sided plot of land was to measure each of the four sides, find the average lengths of the opposite sides, and then multiply the averages. Ancient peoples often thought of area in terms of the boundaries of shapes. One advantage of this method is that all of the needed measurements can be taken without walking across the land itself; however, for many shapes, the method of averaging opposite sides overestimates the area of the shape.

For which shapes would the ancient method produce accurate area measurements? Students will discover that the method works only for squares and rectangles. Even if time is not spent investigating the ancient method, it is still useful to share this connection to the real world with students before continuing to present the task in the following way:

> What if we assume that we can walk across the land, perhaps diagonally from one corner to the opposite corner? Can we come up with a more accurate way to measure the area of four-sided shapes? Today, we will explore the following statement and question: "Sometimes, if I know the lengths of the diagonals of a quadrilateral, then I can find its area. Is this statement true or false?"

The goal at this stage is to agree on what is being asked and to clarify the meaning of the term *diagonal*. Allow students to think about the statement

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TERALS: DIAGONALS AND AREA

for a few minutes and then select students to share their thoughts. Students typically ask questions such as these: "What counts as a diagonal?" "What kind of quadrilateral?" and "Do we know the side lengths, too?" According to the given statement, the quadrilateral could be any size or shape and the side lengths are not known, although it may be possible to find the lengths based on the lengths of the diagonals. Next, ask students what tools they might use to investigate the statement. Protractors and rulers are all that is needed, although software such as GeoGebra could be used.

Working in groups, students should investigate the problem for various quadrilaterals: squares, rectangles, rhombuses, kites, and so on. By the end of the task, each group will have investigated two to four different quadrilaterals, including at least one for which it is possible to determine the area and one for which it is not. It is not necessary that students work through the shapes in a particular order. Plan at least 45 minutes for student investigation in small groups.

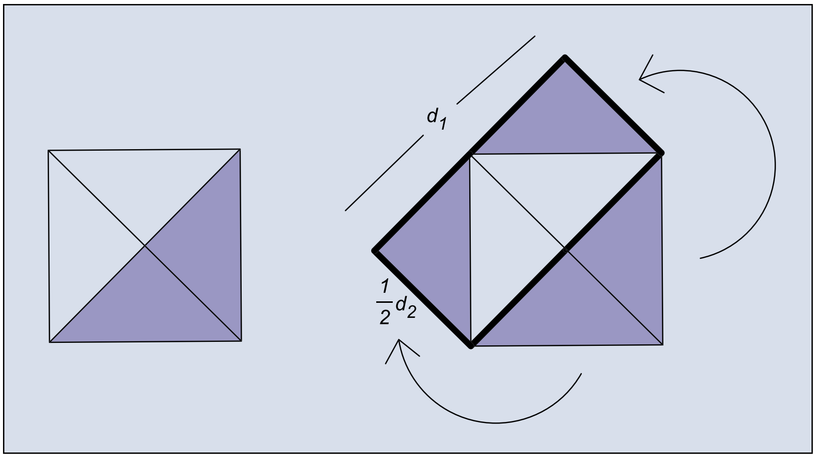


Fig. 1 If I know the lengths of the diagonals of a square, then I can always find the area of the square: *Area* = $d_1d_2/2$.

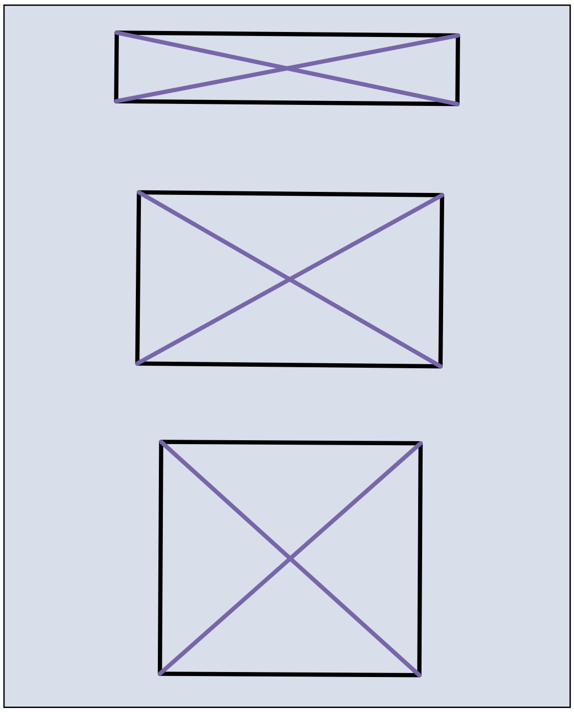


Fig. 2 Noncongruent rectangles may have equal-length diagonals.

Students will need to revisit the definitions of terms such as *rhombus*, *isosceles trapezoid*, and *kite*. Rather than front-loading vocabulary practice, teachers can create opportunities for practice within the mathematical task by encouraging students to look up definitions as needed and by attaching new vocabulary (e.g., *mutually bisecting*) to student thinking. Students will need to be able to calculate the area of a rectangle with specified length and width.

Squares

Drawing a square and its diagonals, students quickly conclude that the diagonals are cut in half where they cross and that the diagonals cross at 90° angles to each other. Students can create arguments for these conclusions by proving that the diagonals cut the square into four congruent 45°-45°-90° triangles. At this stage, students may decide to use a dissect-and-rearrange method to organize the pieces of the square so that the diagonals become side lengths of a rectangle whose area is half the product of the diagonal lengths (see **fig. 1**).

Alternatively, students could use the Pythagorean theorem to determine the side lengths of the square for a given length diagonal, and then multiply to find the square's area. In either case, students discover that, for squares, it is possible to determine area based upon the lengths of the diagonals.

Rectangles

Students initially may spend time attempting to determine the rectangle's side lengths based on a particular diagonal length, which is a useful struggle for them to experience. Students' work provides concrete examples of varying areas that can be shared with the class. Ask each group member to draw, without looking at other group members' work, a rectangle with 10 cm diagonals. Provide scaffolding, if needed, by suggesting that students draw one 10 cm segment and then draw a right triangle using the segment as the hypotenuse. The triangle formed will be half of the rectangle. Asking students to determine the rectangles' areas by measuring the side lengths and multiplying ensures that students realize that different areas are possible.

Students are often surprised that it is possible to draw noncongruent rectangles that have equal-length diagonals (as shown in **fig. 2**); they may believe that the resulting rectangles have equal area or that the diagonals must not be the same length.

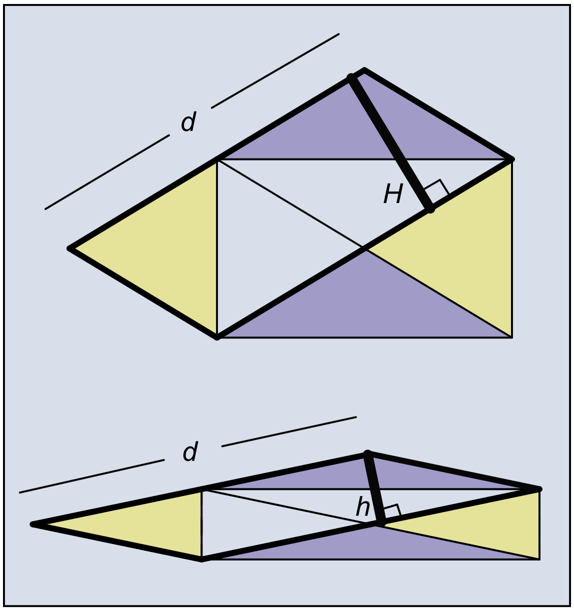


Fig. 3 If I know the lengths of the diagonals of a rectangle, then I cannot always find the area of the rectangle.

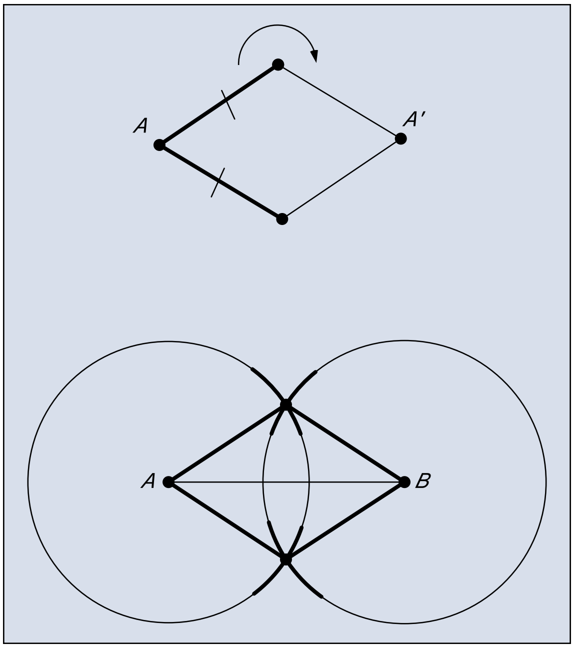


Fig. 4 Giving instruction about how to create a rhombus or providing predrawn shapes may help struggling students avoid unproductive frustration.

Lucas: I think the diagonals are longer in yours. [*points at a long skinny rectangle*]
Angelica: No, they have to be the same; we measured.
Lucas: But your rectangle is longer.
Gerardo: All the rectangles look different. The diagonals are the same, though.
Angelica: The area? I don't know.

Alternatively, students can use the dissect-and-rearrange method to transform the rectangles into nonrectangular parallelograms, each with base d (see **fig. 3**). The parallelograms have the same base, but vary in height and thus in area.

Rhombuses

Unnecessary frustration can lead to fewer opportunities for students to engage with an otherwise rich task. After giving students some initial time for investigation, the teacher might choose to provide scaffolding for creating accurate drawings. For example, a rhombus can be constructed quickly with compass and straightedge or by drawing an isosceles triangle and then reflecting the triangle across its base (see **fig. 4**). Another method for differentiating instruction would be to provide students with predrawn shapes.

Comparing diagonals across differently shaped rhombuses, students discover that the diagonals of the rhombus are perpendicular to each other and are mutually bisecting.

Camila: The lines cross in the middle.
Brooke: Uh huh. And they make a 90.
Camila: We can make four triangles. They are all the same.
Brooke: We can just find the areas and add it up.

It is important at this stage to push for justification. Not only do students need experience formulating arguments for their conclusions, but they also need to come to believe that logical argument is at the heart of mathematics. A justification could be similar to this:

> The two isosceles triangles that form the rhombus are congruent (reflection preserves distance), so the diagonal AA′ must be split

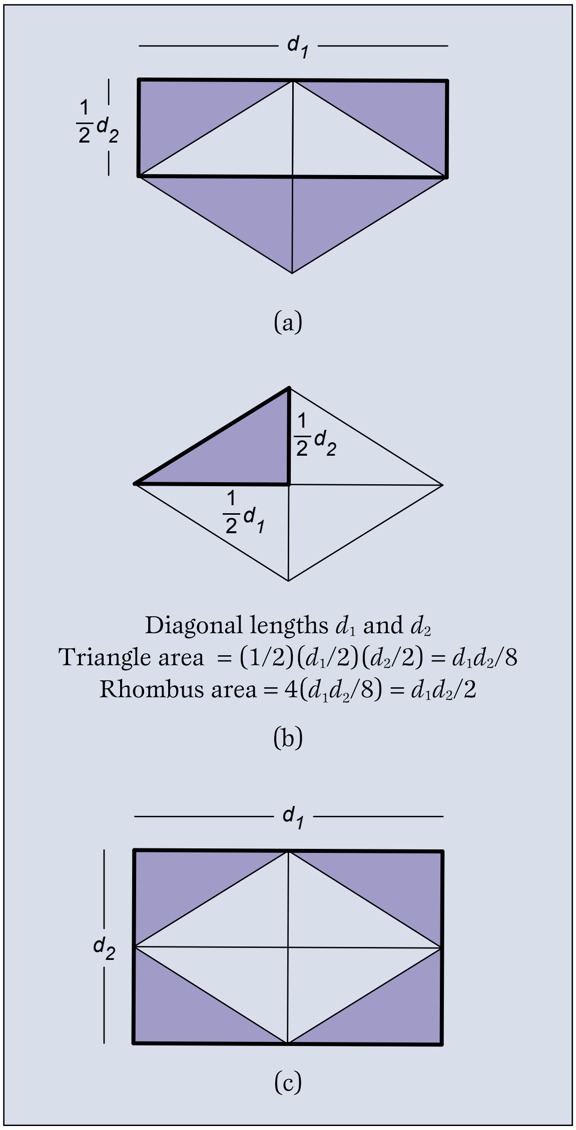


Fig. 5 If I know the lengths of the diagonals of a rhombus, then I can always find the area of the rhombus: *Area* = $d_1d_2/2$.

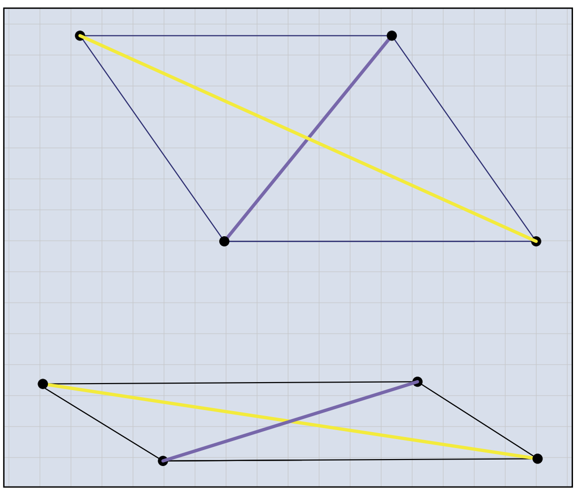


Fig. 6 If I know the lengths of the diagonals of a parallelogram, then I cannot always find the area of the parallelogram.

in half by the reflection line. This is also true if the other diagonal is used as the reflection line, so the diagonals cut each other into equal length pieces.

Once the diagonals have been shown to be perpendicular and mutually bisecting, students can use this information to develop a method of finding the area of a rhombus based on the lengths of the diagonals. They might use the dissect-and-rearrange method **(fig. 5a)** or work directly with a rhombus without rearranging the triangles **(fig. 5b)**.

A third method that students might use to find a formula for the area of a rhombus is surrounding-and-subtracting. Because its diagonals are perpendicular, a rhombus can be surrounded by a rectangle **(fig. 5c)**. The area of the rectangle is the product of the diagonals, and the area of the rhombus is one-half that of the rectangle.

Whichever method is used, students discover that the area of a rhombus is the product of the diagonals divided by two. Because every square is a rhombus, the formula for a rhombus also applies to squares.

Parallelograms and Trapezoids

Investigations of parallelogram and trapezoids (isosceles and nonisosceles) follow the same process as the investigation of a rectangle. For example, the diagonals of a parallelogram are mutually bisecting, but the angle formed by the diagonals can vary; thus, the area can vary as well. Giving specific instructions to students or using predrawn shapes work well; for example, students could be instructed to create two parallelograms, one with sides 8 cm and 10 cm and an included angle of 125° and one with sides 12 cm and 4.5 cm and an included angle of 148° (see **fig. 6**). These parallelograms, with diagonals of equal length, will have different areas.

Kites

Students' intuition accompanied by careful drawings can lead them to conjecture that the diagonals of a kite are perpendicular and that one diagonal bisects the other. An argument proving this conjecture can be developed using triangle congruence and properties of reflections. In **figure 7**, triangles ABD and CBD are congruent (side-side-side), so BD is a symmetry line of the kite and also the perpendicular bisector of segment AC.

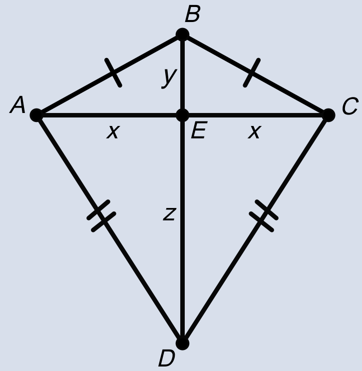
Thinking of the area of the kite as composed of four right triangles, students can then use the dissect-and-rearrange method or the surround-and-subtract method, or define variables **(fig. 7)** to determine a formula for the area of a kite in terms of its diagonals. Students can then consider whether their reasoning and conclusions also hold for nonconvex kites, or darts.

GENERALIZATION AND WHOLE-CLASS DISCUSSION

Once each group of students has thoroughly investigated at least two shapes (one for which an area formula can be derived and one for which it cannot), allot at least thirty minutes for whole-class discussion to share findings and generalize across shapes. When the diagonals are perpendicular, we can always find the area of the shape. When the diagonals are not necessarily perpendicular, we cannot always find the area of the shape. The overarching generalization that the class can arrive at by the end of the discussion is this:

> In addition to diagonal lengths, the angle formed by the diagonals of a quadrilateral must be known in order to determine area.

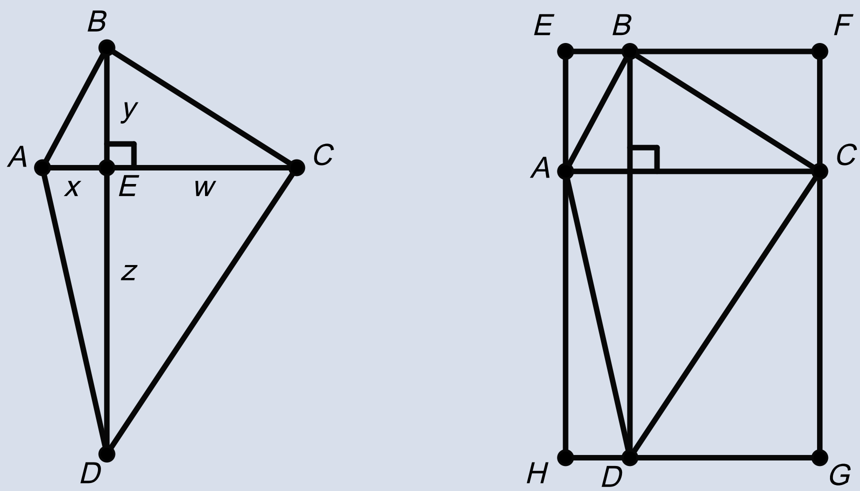
Prior to whole-class discussion, give students time in their groups to write down what they have found and any remaining questions they have. When students discover that other groups may have answers to their remaining questions, the usefulness of whole-class discussion is reinforced. One way to begin the discussion is to describe the goals, namely, to create a class summary of the discoveries made in each group, to decide if mathematical generalizations can be made based on the work of all of the groups, to communicate thinking, and to consider one another's ideas.



$$d_1 = AC = 2x$$
$$d_2 = BD = y + z$$
$$\begin{aligned} Area_{kite} &= \frac{1}{2}xz + \frac{1}{2}xz + \frac{1}{2}xy + \frac{1}{2}xy \\ &= xz + xy \\ &= x(z+y) \\ &= \frac{1}{2}d_1 \bullet d_2 \end{aligned}$$

Fig. 7 If I know the lengths of the diagonals of a kite, then I can always find the area of the kite: *Area* = $d_1d_2/2$.

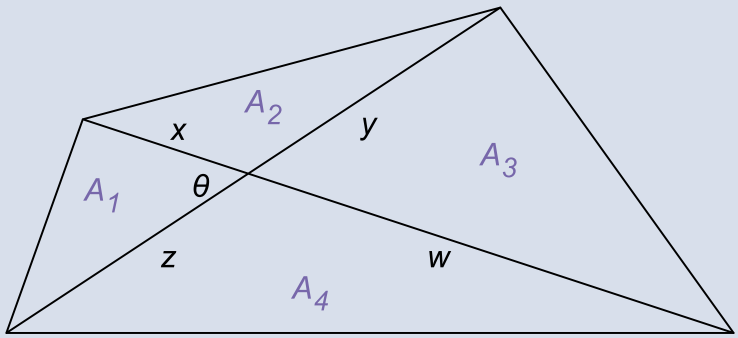
$$d_1 = AC \qquad d_2 = BD$$



$$\begin{aligned} area(ABCD) &= area(\triangle ABC) + area(\triangle ADC) \\ &= \frac{1}{2}(x+w)\bullet y + \frac{1}{2}(x+w)\bullet z \\ &= \frac{1}{2}(x+w)\bullet(y+z) \\ &= \frac{1}{2}AC \bullet BD \\ &= \frac{1}{2}d_1d_2 \end{aligned}$$

$$\begin{aligned} area(ABCD) &= \frac{1}{2}area(EFGH) \\ &= \frac{1}{2}AC \bullet BD \\ &= \frac{1}{2}d_1d_2 \end{aligned}$$

Fig. 8 Multiple strategies can be used to find an area formula for an irregular quadrilateral with perpendicular diagonals.



For any convex quadrilateral, divided into four triangles, with

$$\begin{aligned}
&d_1 = x + w,\ d_2 = y + z,\\
&A_1 = (1/2)\ xz \sin\theta,\\
&A_2 = (1/2)\ xy \sin(180° - \theta) = (1/2)\ xy \sin(\theta),\\
&A_3 = (1/2)\ yw \sin\theta,\\
&A_4 = (1/2)\ wz \sin(180° - \theta) = (1/2)\ wz \sin(\theta);
\end{aligned}$$

We find

$$\begin{aligned}
area_{\text{quad}} &= A_1 + A_2 + A_3 + A_4\\
&= (1/2) \sin\theta\ (xz + xy + yw + wz)\\
&= (1/2) \sin\theta\ [x(z + y) + w(y + z)]\\
&= (1/2) \sin\theta\ (x + w)(y + z)\\
&= (1/2) \sin\theta\ d_1 d_2.
\end{aligned}$$

Fig. 9 The area of a convex quadrilateral is determined by the lengths of its diagonals and the measure of the angle formed by the diagonals. Subtraction of areas would be needed for a nonconvex quadrilateral.

To engage students in critically examining one another's claims, gather the various findings on the whiteboard without judging their correctness with respect to the mathematics itself or the way a finding is stated. Ask students to spend approximately ten minutes in their groups deciding whether they would agree or disagree with each statement and why. Students might also consider whether they think a statement would benefit from being reworded in some way.

EXTENSIONS BASED ON PERPENDICULAR DIAGONALS AND TRIGONOMETRY

The discussion can be extended with these questions: "Is it possible for a quadrilateral to have perpendicular diagonals and not be a kite? If so, can I find its area based only on the lengths of the diagonals?" Students can draw a pair of perpendicular diagonals in such a way as to create an irregular quadrilateral and, based on their previous work, attempt to find an area formula. Two methods for determining the area are shown in **figure 8**.

Some knowledge of trigonometry—specifically, (1) the definition of the sine of an acute angle θ in a right triangle as the ratio of the length of the side opposite the angle and the length of the hypotenuse, and (2) the identity $\sin\theta = \sin(180° - \theta)$ for a given value θ in degrees—allows additional extension. With this information, students can find a formula for the area of a triangle in terms of two sides and the sine of the included angle. They can then use this formula to find a formula for the area of any quadrilateral in terms of the lengths of the diagonals and the angle formed by the diagonals **(fig. 9)**. When the diagonals are perpendicular, the included angle measures 90°, and the more general area formula reduces to $(1/2)d_1d_2$.

A SETTING FOR ACTIVE PARTICIPATION

Geometry students need rich, engaging tasks that build knowledge while allowing for review of vocabulary, measurement, and shape relationships. The quadrilateral diagonal task provides opportunities for students to apply important principles and results, such as the Pythagorean theorem or the equivalence of areas by dissection; solve challenging problems; use triangle relationships and congruence conditions to justify conclusions; develop robust mental images of shapes by examining nonstandard representations of quadrilaterals; pose conjectures; and generalize findings. In addition to these valuable mathematical outcomes, I have found that this task truly engages students and helps to set norms and expectations including active participation in problem solving, reasoning, and communication.

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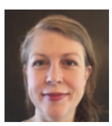

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