INVESTIGATING FUNCTIONS with a Ferris Wheel

A computer activity helps students make sense of relationships between quantities.

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What might students think when they hear the term *function*? A "machine" that takes in inputs and spits out outputs? Perhaps a symbol string, such as $f(x) = x^2 + 1$? A graph shaped like a U? Going further, what might students think when they hear that one quantity is *a function of* another quantity? For example, how might students interpret a statement such as "height is *a function of* distance"?

Teachers can use relationships between changing quantities to help students make sense of function. From this perspective, a *function* refers to a special kind of relationship between quantities. The phrase *a function of* describes a relationship between the quantities.

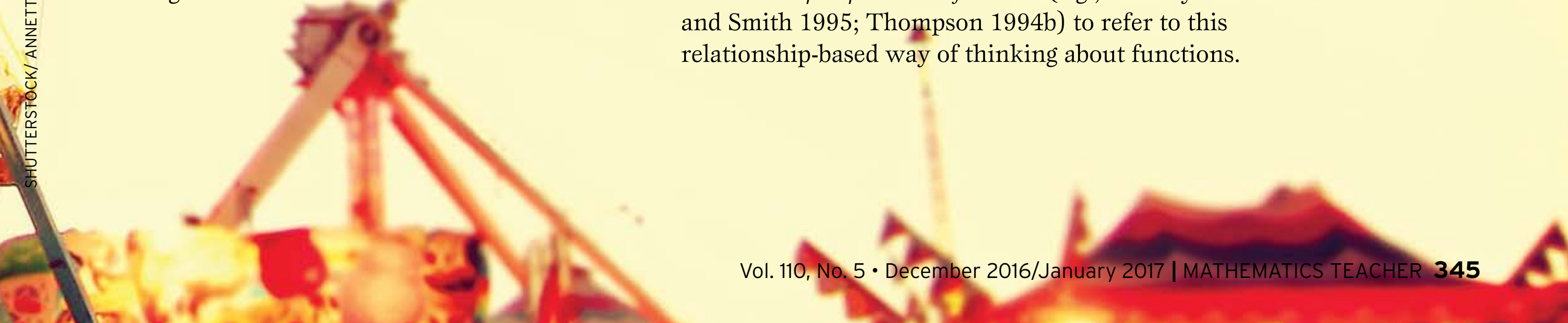
We provide a dynamic Ferris wheel computer activity that teachers can use as an instructional tool to help students investigate functions. We use a student's work to illustrate how students can use relationships between quantities to further their thinking about functions.

FERRIS WHEELS AND FUNCTIONS

Imagine the seats, or cars, of a turning Ferris wheel traveling along their circular path. Could you predict the height from the base of a car to the ground if you knew the distance the car had traveled within one revolution of the wheel? Could you predict the distance a car had traveled within one revolution of the wheel if you knew the height from the base of the car to the ground?

Teachers can ask questions like these to help students use relationships between quantities to investigate functions. In this Ferris wheel situation, students can determine a unique height for any given distance. In contrast, students cannot determine a unique distance for any given height.

Broadly, a function expresses a special kind of relationship between quantities. Chazan (2000) describes the relationship in this way: "Functions are relationships between quantities where output variables depend unambiguously on input variables" (p. 84). Researchers have used the phrase *covariation perspective on function* (e.g., Confrey and Smith 1995; Thompson 1994b) to refer to this relationship-based way of thinking about functions.



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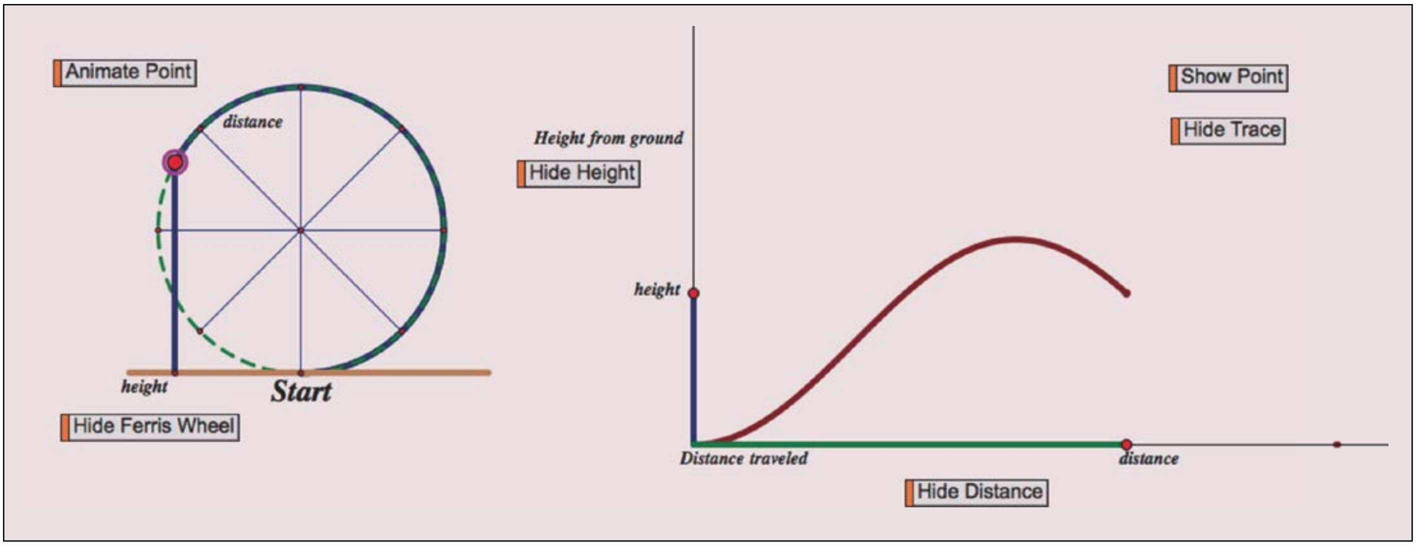

Fig. 1 A dynamic Ferris wheel computer activity relates height and distance.

For the Ferris wheel situation, we can use the term *function* to describe a special relationship between the quantities of height and distance. Specifically, height is a function of distance. Distance, however, is not a function of height. Put another way, height depends unambiguously on distance, but distance does not depend unambiguously on height.

By *quantity*, we mean more than just a label for a unit, such as inches or feet. We mean some "thing" that students can conceive of as being possible to measure (Thompson 1994a). For example, a student could conceive of the height from the ground to the base of a Ferris wheel car as something that she could measure by using a fixed distance between her thumb and forefinger.

USING AN INTERACTIVE COMPUTER ACTIVITY TO INVESTIGATE FUNCTIONS

Johnson used Geometer's Sketchpad® to design a dynamic Ferris wheel computer activity. (These files are available at **www.nctm.org/mt** as more4U content.) The activity links an animation of a turning Ferris wheel to dynamic graphs relating the quantities of height and distance (see **fig. 1**). When students press the Animate Point button, the car (represented by the red dot, **fig. 1**, left) moves in a counterclockwise direction around the Ferris wheel. As the car moves around the Ferris wheel, the linked graph changes dynamically.

The car moves at a constant rate, which would not happen on an actual Ferris wheel ride. To vary the rate at which the car moves, students can click and drag the car to control the motion. In addition, students can speed up or slow down the animation. By design, the dynamic graph represents only one revolution of the Ferris wheel so that students do not also have to keep track of the number of revolutions of the wheel.

When students work with graphs, it is useful for them to think about variation in individual quantities. Thompson (2002) recommends that a student use his finger as a tool to represent variation in individual quantities. Moving a finger horizontally or vertically, a student can track how individual quantities are changing with respect to passing time. In the dynamic Ferris wheel computer activity, students can manipulate and view the dynamic segments, shown on the horizontal and vertical axes (**fig. 1**, right) to represent how the height and distance will change.

Figure 1 shows all elements of the dynamic Ferris wheel computer activity (the Ferris wheel animation, the dynamic graph, and the dynamic segments). Teachers can hide or show different elements to vary students' opportunities for exploration.

Johnson also designed another version of the activity, which represents the height on the horizontal axis and the distance on the vertical axis (see **fig. 2**). By varying which quantities each axis represents, a teacher can provide additional opportunities for student exploration (see also Moore, Paoletti, and Musgrave 2013).

STUDENTS INTERACT WITH THE DYNAMIC FERRIS WHEEL

Johnson implemented the computer activity with a small group of ninth-grade students in an introductory algebra course. Most students began by reasoning about individual quantities of height and distance as changing with respect to passing time. We found that working with the dynamic Ferris wheel computer activity helped students form and interpret relationships between the changing quantities of height and distance.

We share the work of Ana, who interacted with the dynamic Ferris wheel computer activity in

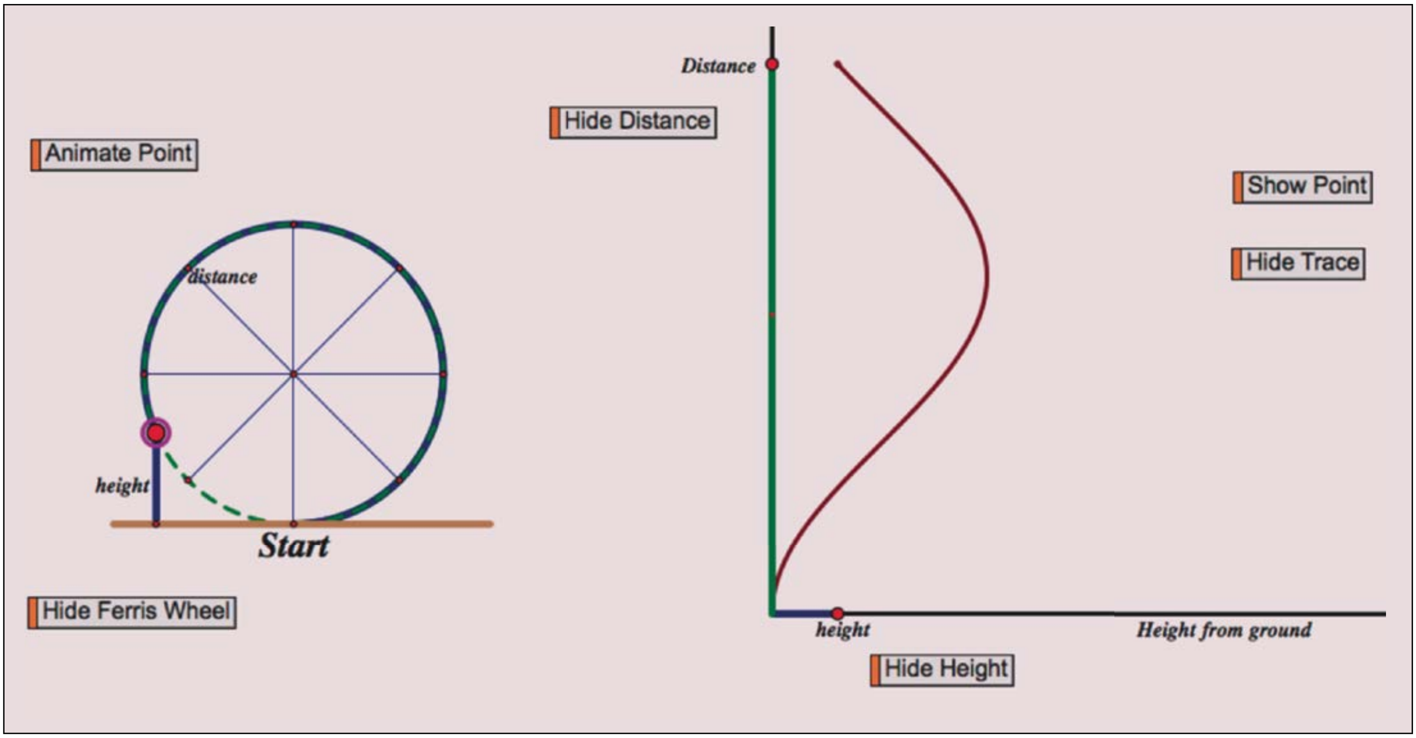

Fig. 2 A second dynamic Ferris wheel computer activity displays the height-distance relationship differently.

individual and small group sessions that were led by Johnson. Ana's work demonstrates the range of reasoning of all the students. We include Ana's work from four different sessions in which she investigated relationships between the changing quantities of distance and height.

Session 1: Height and Distance Changing Separately

After students had seen only the Ferris wheel animation, Johnson asked them to sketch a graph relating a car's height from the ground and its distance traveled within one revolution of the Ferris wheel. **Figure 3** shows Ana's graph.

Notice how Ana drew two graphs on the same pair of axes, one graph for distance and one graph for height. Also notice where Ana placed her labels. Rather than labeling the axes, she labeled each graph.

Ana drew a vertical line extending through both graphs to represent when the car was at the highest point on the Ferris wheel. Although she knew that to reach the highest point the car would travel half

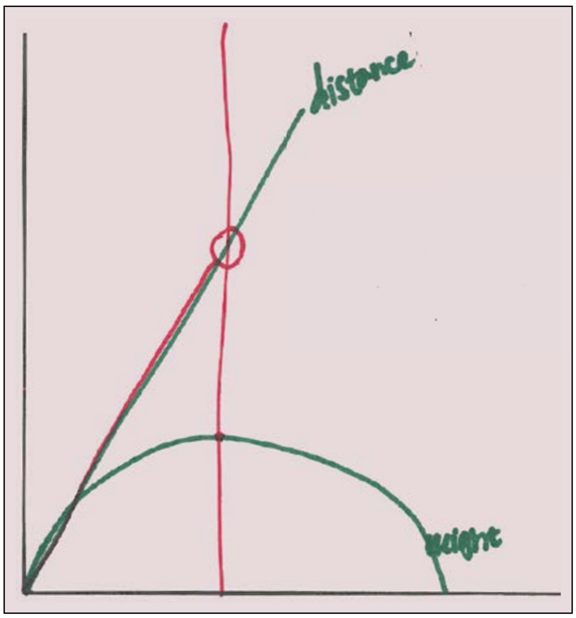

Fig. 3 In her first graph, Ana shows height and distance changing separately.

First, students predicted how each segment would change as the car moved around the Ferris wheel; then they used the dynamic graphs to confirm their predictions.

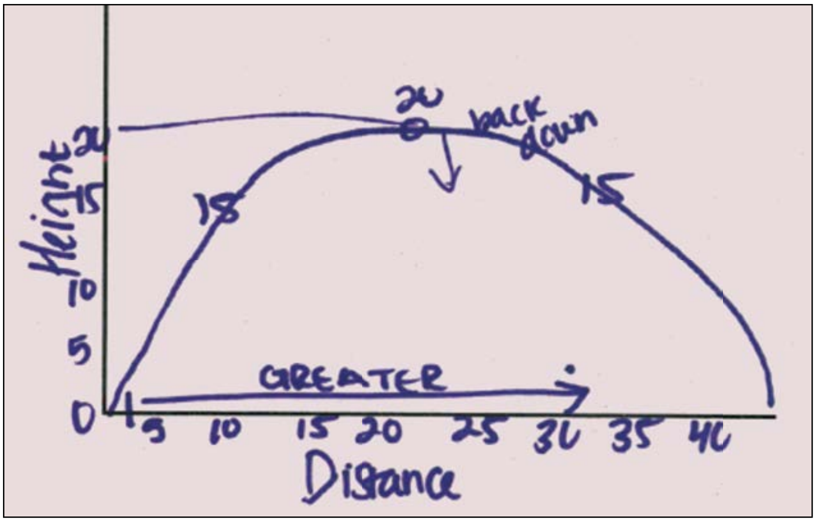

Fig. 4 Axis labels can give teachers insight to students' thinking.

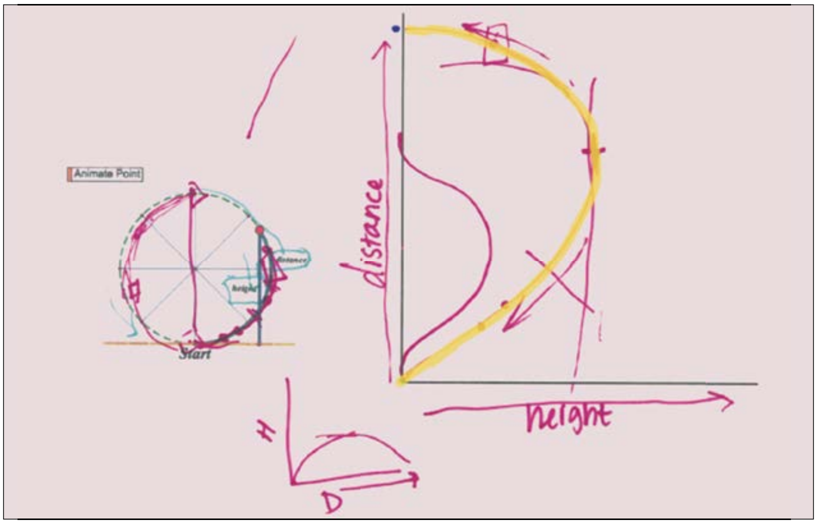

Fig. 5 In her third graph, Ana reversed the axes.

Viewing the dynamic graph, she noticed a new feature—the curvature—that surprised her. She was beginning to develop a more nuanced understanding.

the distance around the Ferris wheel, her graph shows the car traveling far more than half the distance.

Next, students had opportunities to interact with the dynamic segments on the vertical and horizontal axes, representing height and distance, respectively. First, students predicted how each segment would change as the car moved around the Ferris wheel; then they used the dynamic graphs to confirm their predictions.

After making predictions, Ana viewed the dynamic graph shown in **figure 1**. Despite seeing a different graph, Ana did not make any changes to her graph (**fig. 3**). Ana was focusing on how the individual quantities of height and distance were changing with respect to passing time, which her graph represented.

Session 2: Height and Distance Changing Together

In this session, Ana described the changing height and distance in this way: "Distance is greater, greater, greater, and the height is greater, then it stops, and it goes back down." Johnson asked Ana to sketch a graph that represented the distance getting greater, and the height going up and then back down. **Figure 4** shows Ana's new graph.

Notice how Ana's new graph included labels for the height and distance on the vertical and horizontal axes. In addition, she included written descriptions on the graph (e.g., "greater," "back down") to describe how the distance and height were changing.

Although Johnson did not ask Ana to do so, Ana included numbers on the axes before sketching her graph. She did not work from the numbers when sketching her graph, however. Rather, she began at the origin, then sketched the graph in one continuous motion, moving from left to right.

The shape of Ana's new graph (**fig. 4**) looks similar to that of the graph she drew for height, shown in **figure 3**. However, her new graph (in **fig. 4**) represents a relationship between height and distance. Her labels and descriptions show evidence of this.

Session 3: Height and Distance Changing Together

In this session, Johnson again asked Ana to sketch a graph representing the changing height and distance, before viewing a dynamic graph. This time the vertical axis represented distance and the horizontal axis represented height on Ana's graph.

First, Ana drew the highlighted part of the graph shown in **figure 5**. Rather than trying to reflect or rotate the shape of one graph to create a new graph, Ana used the changing distance and height. Ana

Table 1 Ana's Representations of Changing Quantities			
	Quantities Changing Separately	Quantities Changing Together	
Ana's Focus	Individual quantities changing with respect to passing time	Direction of change (increasing or decreasing)	Change occurring in an interval in which one quantity is increasing (or decreasing)
Graph Features	Labels on graphs, rather than axes (**fig. 3**)	Use of a single type of curvature (**fig. 4**)	Use of different types of curvature (**figs. 5, 6**)

drew arrows near the axes to represent the changing distance and height.

This time after Ana viewed the dynamic graph shown in **figure 2**, she noticed a new feature of the graph—the curvature. She said that how the graph "began" surprised her, and then she sketched the smaller, inner graph with different curvatures.

At this point, Ana was not sure why the dynamic graph curved the way that it did. However, she was beginning to develop a more nuanced understanding of the relationships between the changing height and distance.

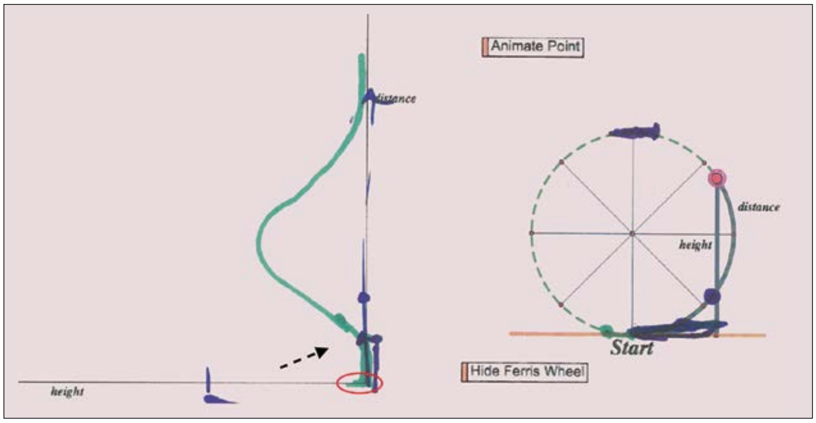

Fig. 6 Ana's fourth graph investigates curvature.

Session 4: Height and Distance Changing Together

In this session, Johnson asked Ana to sketch a graph relating height and distance, with distance on the vertical axis and height on the horizontal axis. This time, the positive direction for height extended to the left. **Figure 6** shows Ana's graph.

Ana's graph in **figure 6** shows a relationship between the changing quantities of height and distance. Furthermore, it includes different curvatures to make distinctions between the ways in which the height and distance are changing together.

Before sketching this graph, Ana stated that she noticed that, at first, the car had begun to move around the Ferris wheel but that the height was still "about the same as where you started." When Johnson asked Ana why her graph showed that, she focused on the lower right part of the graph (see arrow in **fig. 6**). Ana drew a vertical segment to represent how the distance was "still going." Next she drew a small horizontal segment (circled) to represent how the height was "still right here."

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REPRESENTING CHANGING QUANTITIES

Ana's work illustrates three different ways in which students might use graphs to represent changing quantities, in this case distance and height. **Table 1** makes distinctions between Ana's focus on quantities as changing separately or together, and highlights features of graphs she drew when representing the changing distance and

Students who used relationships between quantities to investigate functions could move flexibly between covariation and correspondence perspectives.

height shown in the dynamic Ferris wheel computer activity.

Students often encounter graphs that represent two quantities (e.g., height and distance), neither of which is time. However, a student who is thinking about an individual quantity (or individual quantities) changing with respect to passing time may sketch a graph similar to Ana's first graph (**fig. 3**).

When students begin to use graphs to represent relationships between quantities, they may notice direction of change. For a Ferris wheel, for example, the height increases then decreases even as the distance continues to increase. A student who is thinking in this way may sketch a graph similar to Ana's second graph (**fig. 4**).

As students develop proficiency in using graphs to represent relationships between quantities, it is useful for them to focus on change occurring in an interval in which one quantity is increasing (or decreasing). For example, for the Ferris wheel, consider the interval in which the distance increases from zero to half the total distance. In this interval, the height increases slowly at first, then more quickly, then more slowly as the car reaches the maximum height. A student who is thinking in this way may sketch a graph similar to Ana's fourth graph (**fig. 6**).

VARIATION!

When students are studying function, it is important for them to think about quantities changing together. By interacting with the dynamic segments in the Ferris wheel computer activity, students have opportunities to explore how the individual quantities, height and distance, are changing with respect to passing time.

Once students demonstrate evidence that they are thinking about quantities changing together (e.g., by sketching graphs such as Ana's second graph), teachers can vary the representation of the quantities. For example, we reversed the axes on which we represented height and distance to help focus on changes in an interval in which one quantity is increasing or decreasing.

Although Ferris wheel problems are often used to introduce students to trigonometric functions, we recommend using this context much sooner, for students just beginning to study function. Students can benefit from opportunities to explore situations involving varying rates of change in conjunction with or prior to exploring linear relationships (e.g., Stroup 2002). Because the Ferris wheel activity incorporated varying rates of change, students had the opportunity to investigate change occurring in an interval in which one quantity was increasing (or decreasing).

A covariation perspective is not the only perspective that students should use when studying function. A correspondence perspective is also important, because students should understand that for a function, each input value has a unique output value. In fact, Ellis (2011) found that students who used relationships between quantities to investigate functions could move flexibly between covariation and correspondence perspectives.

Forming and interpreting relationships between changing quantities can provide a foundation for students' understanding of functions. When students have opportunities to think about quantities changing together, they can begin to use a covariation perspective on functions. The dynamic Ferris wheel computer activity is an example of one tool that teachers can use to foster students' thinking about quantities changing together.

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Geometer's Sketchpad files (.gsp) to demonstrate related measurements and distances are available online. This more4U content, an additional benefit, is for members only.