Engaging with CONSTRUCTIVE AND

Logical arguments use examples and existence to prove or disprove four statements.

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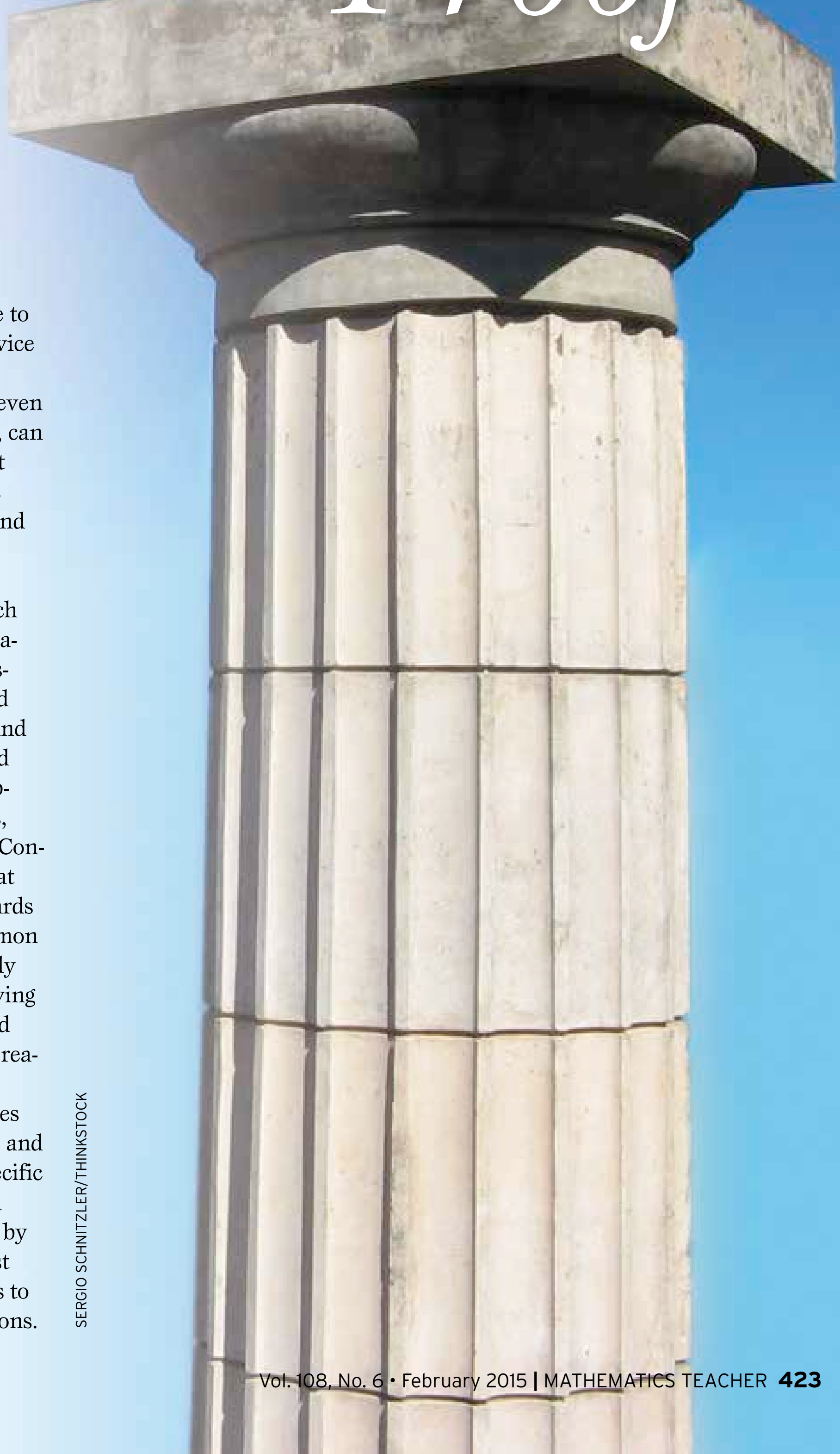
Conjecture and proof are the twin pillars of mathematics. . . . The concept of proof . . . brings something to mathematics that is missing from the other sciences . . . mathematicians have ways to build a logical argument that pins the label of "true" or "false" on practically any conjecture.

—Peterson 1988, pp. 217–18

One method of proof is to provide a logical argument that demonstrates the existence of a mathematical object (e.g., a number) that can be used to prove or disprove a conjecture or statement. Some such proofs result in the actual identification of such an object, whereas others just demonstrate that such an object exists. These types of proofs are often referred to as constructive and nonconstructive, respectively.

NONCONSTRUCTIVE *Proof*



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In this article, we share four tasks that we use to encourage secondary school students and preservice mathematics teachers to consider the conditions under which an example or counterexample, or even the logical demonstration that an example exists, can serve as a proof. We have regularly observed that students and others working through these tasks expand their approaches to proving statements and solving nonroutine mathematical problems.

The use of these tasks supports NCTM's Reasoning and Proof Standard for Grades 9–12, which includes recognizing reasoning and proof as fundamental aspects of mathematics; making and investigating mathematical conjectures; developing and evaluating mathematical arguments and proofs; and selecting and using various types of reasoning and methods of proof (NCTM 2000, p. 342). Also supported are aspects of the Number and Operations, Algebra, and Data Analysis and Probability, and Connections Standards. Moreover, we have found that solvers engaging in these tasks use several Standards for Mathematical Practice (SMPs) from the Common Core State Standards for Mathematics, particularly making sense of problems and persevering in solving them; reasoning abstractly and quantitatively; and constructing viable arguments and critiquing the reasoning of others (CCSSI 2010, pp. 6–8).

These four tasks can be classified into two types on the basis of how they can be resolved. Tasks 1 and 2 can be resolved by finding or constructing a specific example or counterexample that proves the given statement, whereas tasks 3 and 4 can be resolved by showing that an example or counterexample must exist, even if it is not constructed. We ask readers to try resolving these tasks before reading the solutions.

Engaging with the tasks will give a better appreciation for the various solution strategies.

CONSTRUCTIVE-PROOF TASKS

Statements can be proved or disproved with an example.

Task 1: Batting Averages and Simpson's Paradox

This task may be stated as follows:

> Consider two baseball players, A and B. In the first half of the season, player A's batting average was higher than player B's batting average. During the second half of the season, player A's batting average was higher than player B's (again). Prove or disprove that for the entire season player B's batting average can be higher than that of player A. (Note: A batting average is calculated by dividing the number of hits by the number of at-bats; walks are excluded.)

This task involves a well-known statistical phenomenon, Simpson's paradox, which may not be novel to teachers but is not always introduced in secondary school mathematics curricula. The paradox sometimes arises when dealing with aggregate rate data or weighted averages, the latter of which is a standard topic in algebra courses and is usually addressed with mixture tasks, motion activities, and grade-point-average calculations.

Secondary school students are intrigued by the counterintuitive nature of this task. Most, at first, believe that it is not possible for player B to have the higher batting average, but many also feel that the task would be too easy if that were the case. Some solvers try to prove algebraically that it is not possible, often failing to consider the nature of batting averages. For example, **figure 1** shows the work of a preservice teacher who attempted to use a proof-by-contradiction argument but incorrectly summed ratios as fractions.

However, students who understand that players' batting averages are ratios search for at least one combination of hits and at-bats to prove that the scenario is possible. These solvers realize that the half-season batting averages cannot all be based on the same number of at-bats; thus some have more weight than others in the whole-season averages. By cleverly manipulating the number of hits in relation to at-bats, they find appropriate batting averages to show that player B can have the higher full-season batting average, as is seen in a student's solution using this approach (see **fig. 2**).

This high school junior's first reaction to the task was, "They can't have the same number of at-bats." When asked why not, she replied, "Sometimes if I went 6 for 10, a teammate would say she hit better by going 2 for 3 because it's a higher average . . . you really can't compare those because you don't always go 2 for 3." When asked why she used 5 at-bats for batter A in the first half but 25 at-bats in the second half, this student explained, "I wanted to make [batter A's] first average go down a little. The 2 hits in 5 at-bats in the first half didn't do too much . . . they aren't strong enough compared to 25 at-bats." Then she added, "The 200 at-bats [for batter B] have more weight." From her experience as a softball player, this student had a feel for how weighting can affect overall batting averages, and she used this understanding in constructing her example. When debriefed about how she combined the batting averages for the season, she said, "This is not like adding fractions . . . you are adding totals for hits and totals for at bats, then dividing."

Figure 3 shows the work of a preservice teacher who first showed that "typical math" does not work here and then gave a combination of averages that solved the problem.

Students and preservice teachers provide and explain a range of solutions, and we ask them to describe common features of their solutions. This task generates class discussion pertaining to the meanings of fractions and ratios, the idea of weighted averages, and the nonintuitive context of the situation. This task supports NCTM's Number

"No, this is not possible. Let players A and B's batting averages for the first half of the season be denoted A_1 and B_1, respectively, and for the second half of the season, denote the averages as A_2 and B_2. So we know that $A_1 > B_1$ and $A_2 > B_2$. The problem asks, Is it possible for $B_1 + B_2 > A_1 + A_2$? Consider: $B_1 + B_2 > A_1 + A_2 > A_1 + B_2$, because $A_2 > B_2$, so $B_1 > A_1$. This is a contradiction; we know already that $B_1 < A_1$. So we have showed via contradiction that this is not possible."

Fig. 1 A preservice teacher's argument shows incorrect reasoning.

	A	B
1st half	.400 (2-5)	.390 (78-200)
2nd half	.360 (9-25)	.350 (70-200)
Whole Season	$.36\overline{6}$ (11-30)	.370 (148-400)

Fig. 2 A high school student offers a solution by example.

and Operations Standard, which states that students should be able to "judge the reasonableness of numerical computations and their results" (NCTM 2000, p. 393). As the student work suggests, judging the reasonableness of computations and results is critical to resolving this proof. Likewise, three Common Core SMPs (CCSSI 2010) are embodied in the students' perseverance in solving the problem, reasoning through the "nonstandard" operations needed to calculate the batting averages, and their ability to justify their conclusions and communicate them to others.

Note that Simpson's paradox can be observed with actual data, as shown in the batting statistics for Derek Jeter and David Justice during the 1995 and 1996 baseball seasons (Ross 2004, pp. 12–13) (see **table 1**). In both 1995 and 1996, Justice had a higher batting average than Jeter, but when the two baseball seasons are combined, Jeter has a higher batting average than Justice.

Task 2: The Designing Dice Problem

Task 2 is worded as a question, but it can also be posed as a proof task. As a question, it can be answered by constructing an appropriate example. The task is stated as follows:

> If there are no restrictions on the numbers you can place on a pair of cube-shaped dice, is it possible to create a pair of dice such that you can roll all sums from 1 through 12—and only those sums—with equal probability?

We have given this task to middle school, high school, and undergraduate students and often get similar reactions. Some quickly respond that it is not possible to roll a sum of 1; these students have not kept in mind the condition "no restrictions on the numbers" on the dice and are still tied to the idea of standard dice. Others are stymied by the "equal probability" condition. Some initially try a few number combinations haphazardly, while others approach the task more systematically by resorting to their understanding of outcomes, sample spaces, and probability. These latter students realize that if there are 12 possible sums with 36 possible ways to get them, there must be 3 ways to get each sum.

Figure 4 shows the work of a preservice teacher who used this thinking to construct an example. This teacher wrote the numbers as they might appear on a pair of dice. Most students write similar examples either in set notation, as the three shown in **figure 5a**, or in a chart, as the two shown in **figure 5b**.

Although a few students initially do not think that creating such a pair of dice is possible, most of them eventually find at least one set of numbers

" . . . in baseball batting averages, you do not add fractions as you would in typical math:

$$\text{Typical math: } \frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$$

It doesn't make sense that someone could get 7 hits when he has only been up to bat 6 times.

$$\text{Baseball math: } \frac{1}{2}+\frac{2}{3}=\frac{3}{5}$$

In baseball math, you only need to add numerators and denominators straight across. Think of this logically. If [players] came to bat twice during the first half of the season and 3 times in the second half of the season, then they came up for a total of 5 times for the entire season. Using this notion, we can come up with a way to solve this problem.

	Player A		**Player B**
First Half	3 hits/9 at-bats	>	3 hits/10 at-bats
Second Half	1 hit/1 at-bat	>	7 hits/10 at-bats
Whole Season	4 hits/10 at-bats	<	10 hits/20 at-bats

. . . In the end, player A had a total batting average of .4, which is less than that for player B, who had a batting average of .5."

Fig. 3 A preservice teacher correctly uses an example as proof of existence.

Table 1 Data from 1995 and 1996 MLB Seasons That Demonstrate Simpson's Paradox

	1995	**1996**	**Combined**
Derek Jeter	12/48 = .250	183/582 = .314	195/630 = .310
David Justice	104/411 = .253	45/140 = .321	149/551 = .270

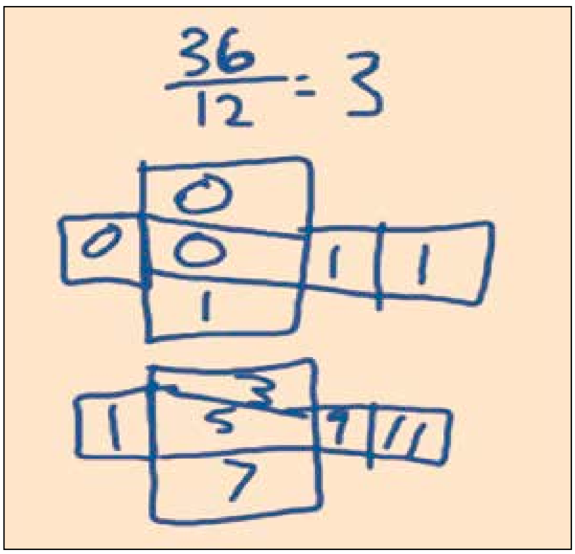

Fig. 4 One preservice teacher constructs a proof of the Dice task.

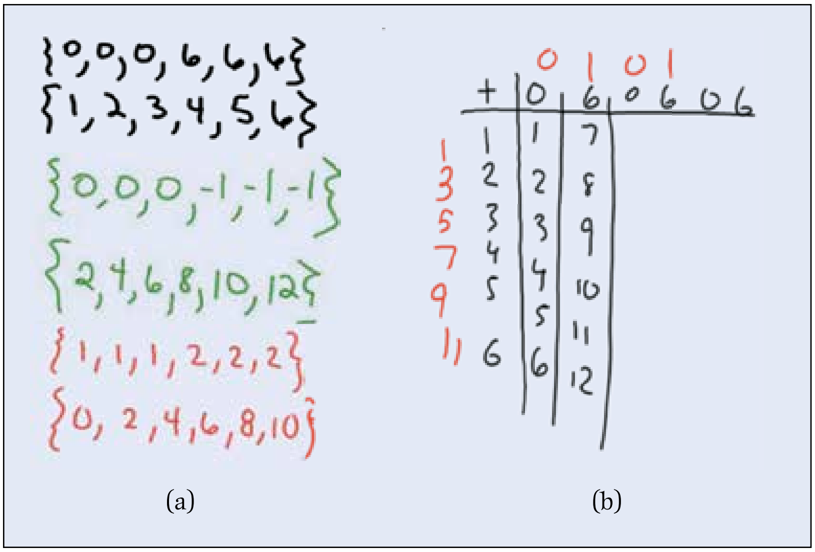

Fig. 5 Two students work to resolve the Dice task.

"At this point, I started looking at more properties of exponents and started looking at this one:

$$\left(a^b\right)^c = a^{bc}$$

I tried using the square root of 2 first and determined the following:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\bullet\sqrt{2}}$$

Here it is important to realize that

$$\sqrt{2}\bullet\sqrt{2} = 2.$$

Thus,

$$\sqrt{2}^{\sqrt{2}\bullet\sqrt{2}} = \sqrt{2}^{2} = 2.$$

Therefore, we see an example of when an irrational number raised to an irrational number is rational under one specific circumstance."

Fig. 6 A proof attempt demonstrates a common flaw.

that satisfy the task conditions and answer the question in the affirmative. Once several solutions are shared publicly, students see that all their solutions involve two numbers each appearing three times on a die, and they realize that there are an infinite number of possible dice that satisfy the conditions. Students connect solutions and observations to the definitions of probability and the notion of sample space, either while finding examples or after observing those found by others.

This task offers a unique way of addressing NCTM's Data Analysis and Probability Standard on two levels. Specifically, this Standard states that students should be able to "understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases" (the first level) and that students should "understand how to compute the probability of a compound event" (the second level) (NCTM 2000, p. 401). Further, when making connections among solutions, observations, and mathematical definitions, students "recognize and use connections among mathematical ideas" (NCTM 2000, p. 402), which is a cornerstone of the Connections Standard. This task also allows middle school students to "investigate chance processes and develop, use, and evaluate probability models" (CCSSI 2010, 7.SP.C, p. 50) while also engaging them in the SMPs.

NONCONSTRUCTIVE-PROOF TASKS

These statements can be proved by demonstrating that an example or counterexample exists.

Task 3: Prove or Disprove: An Irrational Number Raised to an Irrational Power Can Be Rational.

At first glance, students often believe that this statement cannot be proven true. It is not until they consider familiar irrational numbers that solvers reconsider their initial reactions. This task helps students engage in number theory concepts in a unique way. One common pitfall for some students is failure to demonstrate or even state that the numbers they are using in their solutions are in fact irrational. This is a critical step in proving the statement and a significant practice in writing proofs. For example, the student whose work is shown in **figure 6** never even considered that his demonstration requires him to show that $\sqrt{2}^{\sqrt{2}}$ is itself irrational before raising it to an exponent.

Some solvers who are not completely certain about the irrationality of the number $\sqrt{2}^{\sqrt{2}}$ realize that they do not need to do so. If they can use this number to show that there exists at least one rational number that can be written as an irrational number raised to an irrational power, then the statement is proved. For example, the preservice teacher whose work is shown in **figure 7** used the same numbers as the student whose work is shown in **figure 6**, but the logic was very different.

This approach, shown in **figure 7**, considers two cases. Case A (which starts in the left column of work and ends with the top line of the right column) assumes that $\sqrt{2}^{\sqrt{2}}$ is a rational number and thus demonstrates the conjecture. Case B, on the other hand, assumes that $\sqrt{2}^{\sqrt{2}}$ is irrational. In this case, raising it to the irrational power $\sqrt{2}$ leads to

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\bullet\sqrt{2}} = 2.$$

Hence, case B also shows an irrational number raised to an irrational power equaling a rational number (i.e., the number 2). Only one of these cases must be correct, but the preservice teacher did not say which of the two is correct. However, doing so is not necessary to prove the given statement; this logical argument demonstrates that at least one example of an irrational number to an irrational power equaling a rational number exists. Because neither case and, hence, neither example is identified as the correct one, this proof is considered *nonconstructive*.

This task extends the Common Core Standards of Mathematical Content (SMCs) and concerns the meaning of rational exponents (i.e., HSN-RN.A.1) and understanding the sums and products of rational and irrational numbers (i.e., HSN-RN.B.3), by giving students an occasion to engage in the SMPs addressing problem solving, reasoning, and constructing and critiquing arguments (i.e., SMPs 1, 2, and 3). Moreover, the nature of this task gives students an opportunity to "develop an appreciation of mathematical justification" and requires that "their standards for accepting explanations should become more stringent" (NCTM 2000, p. 342). NCTM recommends that as students progress through high school, their level of sophistication with regard to proof increases.

Task 4: Prove or Disprove: In New York City, There Are at Least Two People with the Same Number of Hairs on Their Heads.

We usually present this task orally and often pause after the first three words, to which students respond with a combination of groans, sighs, and "Oh no." Then, after we get past a number of proposed trivial solutions, such as "Mr. Clean and Kojak," students usually start looking for information on the population of New York City and information on the average or typical number of hairs on human heads. Some use their found information to argue that there must be some people with the same number of hairs on their heads in New York City because the number of people in the city (e.g., 8,244,910) is so much larger than the typical number of hairs (or hair follicles) on heads (between 100,000 and 150,000). Most students respond that this is not a proof, but many of them also believe that it is a reasonable and almost convincing argument.

Other students use their found information to make a similar argument, but one based on the "pigeonhole" principle, which many have not formally learned in any mathematics course. For example, the high school student's work shown in **figure 8** suggests thinking consistent with an understanding of the pigeonhole principle.

This student assumed there are 8 million people in New York, found estimates for the typical number of hairs on a human head that ranged from 100,000 to 150,000, doubled 125,000 "to be sure," and reasoned that after one person was found with a hair count for each of the numbers from 0 to 250,000, there would still be 7,750,000 New Yorkers with a hair count equal to a number already used.

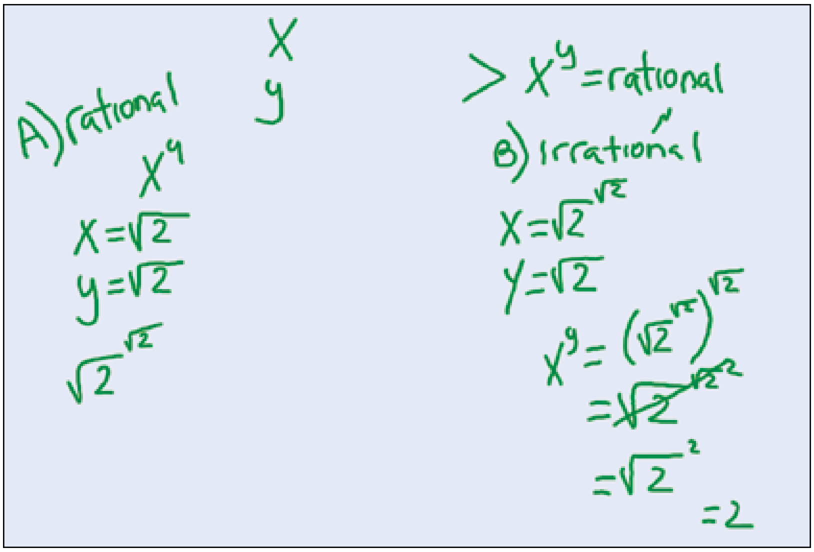

Fig. 7 A preservice teacher considers two cases in a nonconstructive proof.

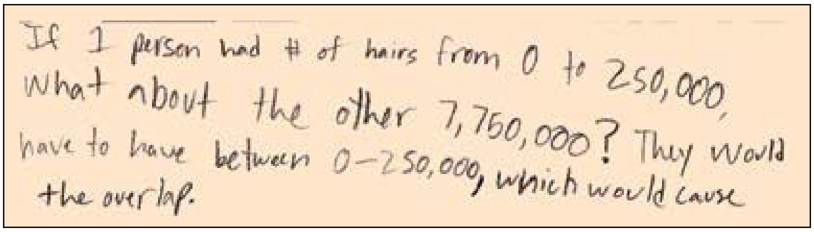

Fig. 8 A student's solution echoes the pigeonhole principle.

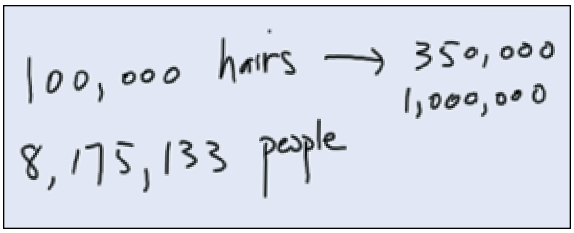

Fig. 9 A preservice teacher's work leads to a pigeonhole proof.

Similarly, **figure 9** shows the work of a preservice teacher who first stated that, on average, humans have 100,000 hairs but "to be safe assume [that] the maximum number of hairs would be 350,000 or even 1,000,000." She then argued that if one tried to sort the more than 8 million people in New York City into different "hair number" categories (pigeonholes), based on the number of hairs on their heads, after a million people were put into unique categories, the other 7 million would need to be placed in at least one of the already-taken categories.

This is another example of a *nonconstructive proof*: Two people with the same number of hairs on their heads are not identified, but it is logically demonstrated that under the given conditions, they must exist. Indeed, a high school student remarked, “I know it’s the case even though I didn’t count the hairs on every person’s head.”

NCTM’s Reasoning and Proof Standard states: “The repertoire of proof techniques that students understand and use should expand through the high school years” (NCTM 2000, p. 345). Task 4 provides students an opportunity to learn a proof strategy that may not be found in a typical high school textbook, a beneficial experience for them when conjecturing and justifying in unfamiliar mathematical situations.

PURPOSEFUL PRACTICE AND THE POWER OF PROOF

There are several points to keep in mind when using these tasks. Although they can be used to help students develop the ability to prove conjectures, if not used cautiously tasks can lead to misconceptions about proof by example. Teachers must be careful to help students see that an example does not and cannot prove many types of conjectures. Students need to thoroughly analyze statements and conjectures to be proved or disproved, carefully considering the meaning and significance of pivotal words such as *is*, *can*, *will be*, *always*, *all*, and so forth. Learning to judiciously analyze tasks like these will help students with other aspects of doing mathematics and other subjects as well.

Also, these tasks can be modified to meet teachers’ specific goals and the needs of particular students. If you want your students to practice writing formal mathematical proofs, have them write up their arguments as such; however, if you want your students to practice mathematical critical thinking, ask them to brainstorm arguments and discuss them in pairs or small groups. In addition, present these tasks with various wordings, such as “Prove or disprove,” “Is this possible?” “Explain when,” or even “Find a case in which this works” (to simplify the tasks by restricting the outcomes).

Davis and Hersh argued that the purpose of generating a proof in mathematics has been for “validation and certification” (1981, p. 149) but added that a proof “increases understanding by revealing the heart of the matter . . . proof is mathematical power . . .” (p. 151). Indeed, the Common Core recognizes learning how to construct and evaluate proofs as an important component of a student’s mathematical development:

> Mathematically proficient students . . . make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. . . . Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. (CCSSI 2010, pp. 6–7)

Thoughtful use of the tasks presented here can help students develop mathematical power and proficiency.

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