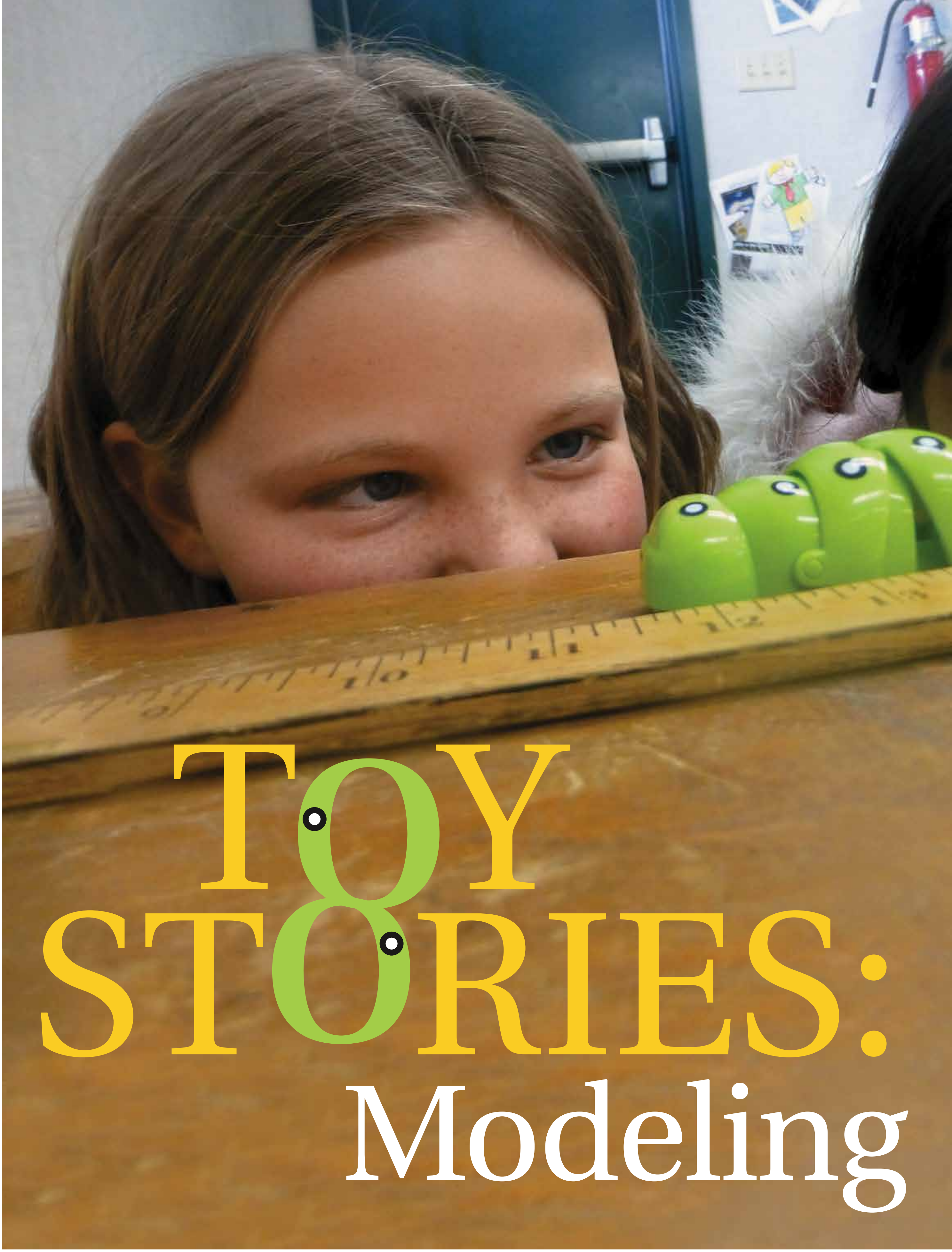

# TOY STORIES: Modeling

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# Rates

Patricia E. Swanson

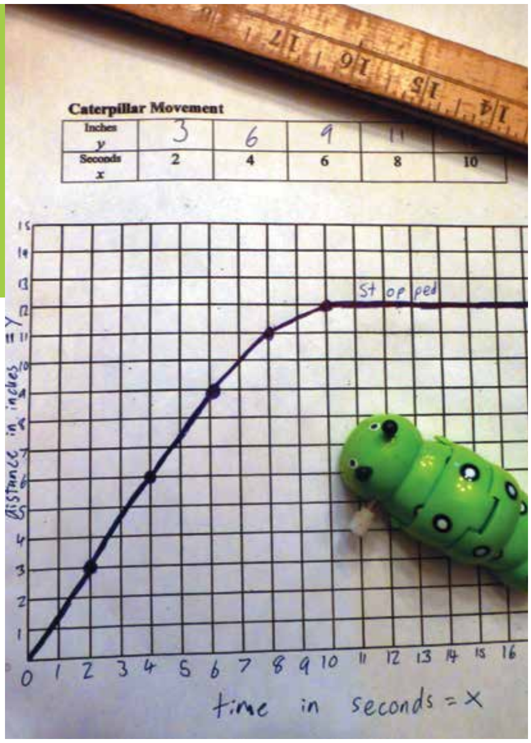
## Using wind-up toys to compare distance over time gave these children a stake in understanding how the numbers make sense.

Elementary school mathematics is increasingly recognized for its crucial role in developing the foundational skills and understandings for algebra. Among the most challenging of these concepts is the relationship between two variables—covariation—as represented by a series of ratios comparing two quantities, paired values on a T-table, or points on a graph. Each of these representations tells a story of two quantities, or measures, and how they are related; but the story is poorly understood by many students (Lobato and Thanheiser 2002).

I explored this issue last year, researching the teaching of critical foundations of algebra (National Mathematics Advisory Panel 2008) in a K–grade 8 one-room rural school. Although I worked with third through eighth graders, this lesson was designed primarily for the fourth through sixth graders as a lead-in to understanding rates and how to show them on a graph.

Rates are a special kind of ratio—whereas ratios compare two quantities in a given situation, rates compare two different types of measures, in different units (Van de Walle, Karp, and Bay-Williams 2010). Comparing the number of miles to hours traveled is a rate because it compares different kinds of measures. Rates often describe how quantities change over time (Lamon 2005). When rates are expressed as a comparison to a single unit, they are called *unit rates*—for example, the number of miles traveled in just one hour, or the cost for a single item (Chapin and Johnson 2006). Students are taught to calculate unit rates by dividing the first term by the second; for example, if a car travels 100 miles in 2 hours, students divide 100 by 2 to calculate the unit rate of 50 miles per hour.

This lesson introduces the concept of rates—comparing two different types and units of measure—and how to graph them.

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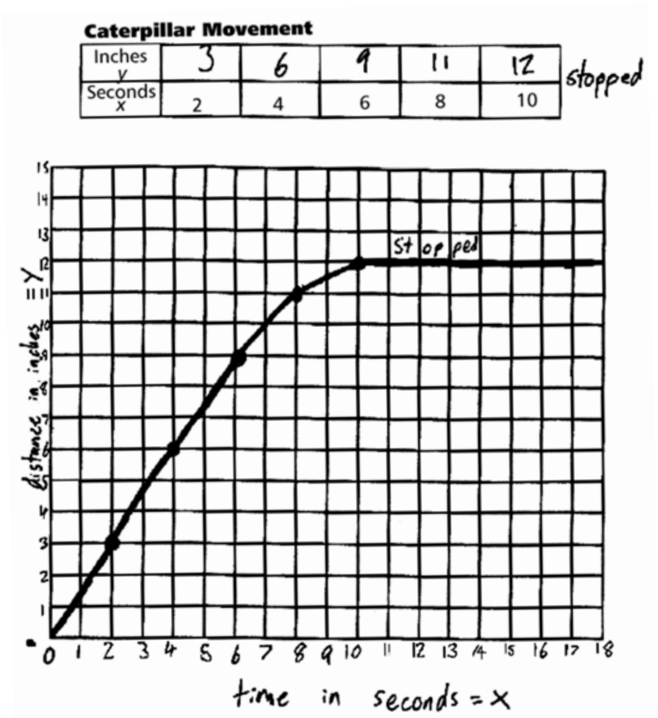
In this lesson, I chose the concept of speed, a rate comparing distance over time, to present students with a concrete, hands-on experience in which they could gather data on two measures, or variables, and explore their relationship. We quickly discovered that the lesson would provide a crucial reference point for an array of subsequent lessons focused on rates, proportional reasoning, graphing, and speed as represented by the slope of a line. But it was one student's question as she struggled to make sense of the numbers that demonstrated the value of the lesson and its potential to engage students in practices central to the discipline of mathematics.

In this article, I describe the lesson and share materials I developed to teach it. I then use the student's question as a lens to discuss the mathematical practices that students engaged in as they attempted to make sense of their data. This lesson illustrates how we can put into place key mathematical practices identified in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010).

FIGURE 1

Taken from a student activity sheet, this table shows a series of caterpillar rates expressed as distance over time. The student then graphed the table data as ordered pairs.




## The lesson

We collected data on the speed of small wind-up toys, measuring distance traveled over time. We discussed that as mathematicians, we could measure speed as a rate, in this case comparing distance measured in inches to time measured in seconds. By measuring the total distance traveled by a toy at a series of specified times, we could represent our data as a series of rates, or organized in a T-table showing $x$- and $y$-coordinates representing time and distance. These coordinates locate points on a line graph. The slope of the line segments connecting the points represents the rate we call *speed*. Each of these representations is a tool we use to model with mathematics, mapping the relationships between quantities to interpret and draw conclusions (CCSSI 2010). I asked students to compare these tools and decide which they believed best modeled their observations of the toys' movement.

We first practiced with a delightfully slow and steady caterpillar toy, recording our data as a series of rates expressed as distance over time (see **fig. 1**). We organized consecutive time measures on a table showing the total distance the caterpillar traveled in two, four, six, eight, and ten seconds. Students were familiar with graphing points on a coordinate grid, so we used our tables to create ordered pairs that could be graphed. We represented time on the $x$-axis and distance on the $y$-axis. As guided practice, each student recorded the caterpillar data in his or her own table and graphed the points showing distance over time. We discussed the line graph, noting that the first three points on the graph reflected the caterpillar apparently traveling at a constant rate of three inches for every two-second interval. However, after six seconds, the toy slowed, and as its speed decreased, the slope of the line segments connecting the next two points changed and became less steep. Finally, after ten seconds, the caterpillar stopped, and the line became horizontal, showing that the toy gained no further distance as time passed. Students noted that our line graph did not look like most line graphs in their textbook, showing one line representing a constant rate. Rather, our line graph was in line segments, the slope of each representing the toy's speed between consecutive time measures. I referred to our graph as *a mathematical representation modeling the toy's progress over time*, but students called it a *toy-story line*—a graph with a story to tell.

I assigned students to work in pairs or triads, giving each group a bag with two or three wind-up toys to compare. Each group had a task card outlining the group project and individual activity sheets on which to record their data and explain their findings (see **online activity sheets 1** and **2**).

FIGURE 2

Measuring time and distance as a series of line segments —rather than of proportional rates as a single line— taught students to look for the context of rate problems.

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## Group work

Although students found the task highly engaging, it carried its share of pitfalls. Students predictably fixated on the toys, not the math. They were determined to find out which toys were fastest by racing them rather than measuring distance over time. In addition, the toys' movement was not always easy to measure. Some traveled quickly; others stalled out or required periodic nudges to travel in a straight line. In retrospect, I should have selected the wind-up toys more carefully. Slow, relatively steady, sturdy toys are best for this activity.

Nonetheless, the challenges of collecting data on toy rates led to rich discussions on the meaning of rates and the importance of understanding them in context (Lamon 2005). The wind-up toys illustrated that rates can be constant or varying, a key point often ignored in the student text. Indeed, "real" data on moving objects, coordinate points showing distance over time, rarely fall on a perfect line. In addition, although most texts at this level focus on extending rates proportionally, that computation would be misleading in this context. Rather than a series of proportional rates modeled by one single line, our line graphs showed a series of line segments (see **fig. 2**). The slopes of the line segments reflect the speed of the toy between two consecutive points in time. Ultimately we would need to bridge to the text and practice calculations assuming constant rates, but this activity set the stage for my students to critically examine the context of any rate problem. In subsequent lessons, when grappling with rate problems, our discussions always began with the meaning of the rate and the context in which it was used.

In spite of our rather rocky beginning, groups ultimately settled into the task. I had purposely designed the task so that students would need to discuss and decide a number of essential issues. For example, at what time intervals would they measure distance? How would they scale their graph? This structured

**FIGURE 3**

Fifth-grade students, trying to convey a wind-up bird's speed, scaled distance on a graph by dividing the greatest distance the bird traveled by the number of squares on the graph.

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**FIGURE 4**

One of the fifth graders taught her younger partner how to skip-count correctly to label the *y*-axis of her graph.

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uncertainty led to productive group conversations (Lotan 2003). Because I structured the groups to be academically heterogeneous, these math discussions also involved considerable helping and teaching. For example, in one group, a fifth grader who was trying to capture the rapid movement of the group's speeding wind-up bird (see **fig. 3**) grappled with how to scale distance on the graph (see **fig. 4**) by dividing the greatest distance the bird traveled by the number of squares on her graph. She then taught her younger partner how to skip-count correctly to label the *y*-axis of her graph.

## Wrap-up

When we gathered as a class, we discussed students' accurate data representations both on tables and line graphs. The graphs showed the toys' progress, or distance, at different points in time, and the lines connecting the points documented the toys' somewhat erratic speed.

As is often the case, the multifaceted mathematical potential of the lesson emerged during our wrap-up session, when we were discussing students' data. First, we used samples from students' data to practice calculating the unit rate or average speed of the toys at given points in time. By dividing the total distance traveled by the total time elapsed, we calculated the distance traveled in just one second. Note that the unit rate or average speed does not reflect (as many students think) the arithmetic mean of the different speeds. Rather, average speed implies *proportional distribution*, as if the toy had traveled at a constant rate for the total elapsed time (Lamon 2005). For example, on the caterpillar graph, the line segments' slopes reflect several different rates, or speeds, but the unit rate, or average speed, for the total elapsed time of ten seconds assumes a constant speed calculated at 1.2 inches per second.

We selected rates from students' data tables to practice. I started with "friendly" numbers. For example, the toy car traveled a total of 6 inches in three seconds; six divided by three gave us a unit rate, or average speed, of 2 inches per second. We practiced with increasingly difficult rates, the older students dividing distance by time to calculate the unit rates, and the younger students checking their answers with calculators. Using student data to calculate unit rates and practice computation gave students a stake in finding meaningful answers.

## Sense making

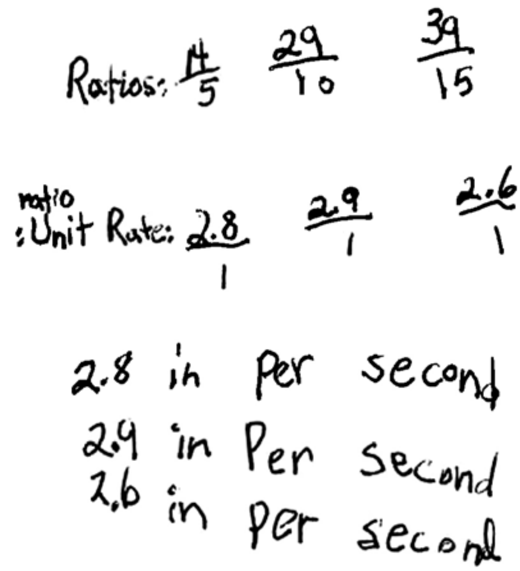
From these computations, an older student, Cecilia, who often asked *how* to do something in mathematics but rarely asked *why,* looked at her data (see **fig. 5**) and stated, This doesn't make any sense. The fractions are getting bigger, but the other, the unit rate—see here? It gets smaller.

Her question illustrates a serious misconception regarding fractions, as it is not the size of the numbers but the relationship of the part (the numerator) to the whole (denominator) that indicates the size of the fraction. Similarly, the relationship between distance and time defines speed. Although understanding this relationship is critical, addressing Cecilia's struggle to make sense of her data was of first importance. Cecilia was engaging in the first of eight mathematical practices identified in the Common Core—to *make sense of problems and persevere in solving them* (CCSSI 2010). Cecilia rarely tried to make sense of mathematics. It was a milestone for her in that on this day, having collected the numbers herself, she thought the numbers ought to make sense.

The National Research Council identified productive disposition as one of the five interrelated strands of mathematics proficiency. They defined productive disposition as the

**FIGURE 5**

When Cecilia's rate computations prompted her to question *why* the numbers made sense, she was engaging in the first of the Common Core's eight Standards for Mathematical Practice.





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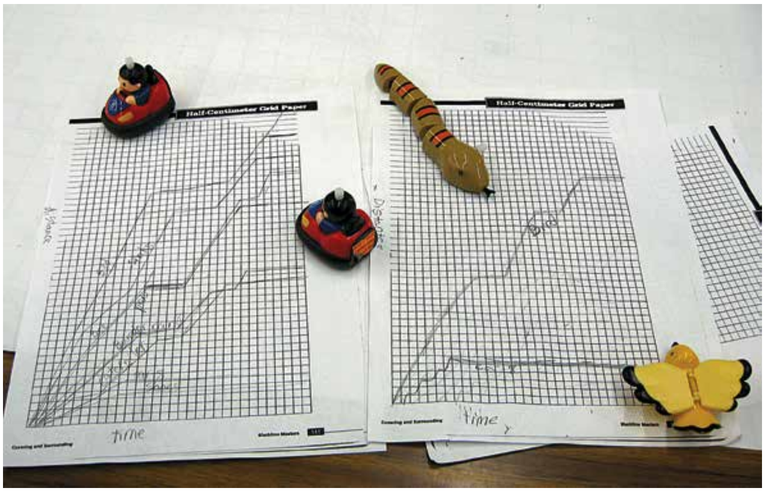
Using data she had collected herself gave Cecilia a stake in making sense of her numbers. Her teacher commended Cecilia's efforts to "think like a mathematician."

FIGURE 6

Students informally observed and mapped different toys' speeds (without numbers), noting that "the steeper the line, the faster the toy." They then used the graphs to describe how to model and interpret each toy's changing speed.



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"habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (Kilpatrick, Swafford, and Findell 2001, p. 116). Cecilia was developing a productive disposition—the expectation that mathematics ought to be sensible and that she was persisting.

I called the class together to explore Cecilia's question, first pointing out that her question showed she was engaging in the most important mathematical practice, to make sense of her data. Her attempt to link the toy's observed movement to the numbers she generated and her expectation that both her "fractions" and her unit rates should tell the same story, were commendable. She was thinking like a mathematician.

We used her question as an opportunity to engage in several other of the Common Core's Standards for Mathematical Practice (SMP) to make sense of the data: (1) to *model with mathematics* (SMP 4), (2) to *look for and express regularity in repeated reasoning* (SMP 8), and (3) to *reason abstractly and quantitatively* (SMP 2). We revisited our data on the caterpillar whose slow, steady movement yielded friendly numbers on the ratio table.

We once again observed the caterpillar's movement and crosschecked our first three rates on the table with points on the graph. I asked students if they could tell me how far the caterpillar went in just one second—not by dividing, but by looking at the number patterns on their tables or the line on the graph. In their groups, they once again discussed the caterpillar data, looking for regularity or patterns in the data that they could use to find the unit rate. Some used the rate table, reasoning that if the bug went 3 inches in two seconds, then it would go 1 1/2 inches in one second. Others noted that the line on the graph showed a distance of about 1 1/2 inches at the one-second mark on the *x*-axis. Finally, we examined Cecilia's question: Why did the numbers on the rate table get bigger when the unit rate at first stayed the same and then decreased? Students engaged in the second mathematical practice, to reason abstractly and quantitatively, as they divided to find unit rates and grappled with the question. They compared the increasing numbers on the ratio table with the initially constant but then declining unit rates for the caterpillar. They explained to one another why the numbers representing time and distance got bigger, but the unit rate, the relationship between the two numbers, did not. Their conversations led to one final example of modeling with mathematics.

Students noted that the steepness of the lines on their graphs reflected the toy's speed. In their words, "the steeper the line, the faster the toy." When the line flattened, the toy was slowing down. A horizontal line meant that the toy had stopped. The changing slopes of the line segments on their graphs modeled the story of the toy's speed over time. I challenged them to informally map—without numbers—the speeds of different toys, estimating the slopes of the lines to illustrate the toys' changing speed. With gusto, students gathered around a large table, observed each different toy travel across the floor, and sketched what its graph might look like. Although their informal line graphs were based on observation, not measurement, I prompted them to describe how each toy's changing speed could be modeled and interpreted with their informal graphs (see **fig. 6**).

## Traveling from concrete to abstract

Measuring rates with wind-up toys was not only fun but also a mathematically rich task. I often use highly engaging, concrete tasks such as this to serve as foundational activities for a unit of study. I believe these tasks merit the time, for they serve as a memorable grounding experience for students, a concrete example, and a reference point for subsequent lessons. If we are to help students travel from concrete to abstract representations, to move from an understanding of whole numbers to the challenges of rational numbers and proportional reasoning, then we must scaffold their understanding with such tasks as this one. It promoted substantive mathematical discussion and presented the opportunity for students to engage in the kinds of mathematical practices that foster deep engagement in the discipline of mathematics.

**Common Core Connections**

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|---|---|
| SMP 1 | SMP 2 |
| SMP 4 | 6.RP.3 |

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Using a timer, measuring tools, and wind-up toys, students compared distance over time in a fun, mathematically rich task.

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