Encouraging Perseverance in Elementary Mathematics:

A Tale of Two Problems

According to professor Alan Schoenfeld, many students hold the following beliefs about mathematics:

- Mathematics problems have one and only one correct answer.
- There is only one correct way to solve any mathematics problem—usually the rule that the teacher has most recently demonstrated to the class.
- Ordinary students cannot be expected to understand mathematics; they simply memorize it and apply what they have learned mechanically.
- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any problem in five minutes or less. (Schoenfeld 1992, p. 359)

Schoenfeld claims that students extract their beliefs about formal mathematics in large measure from their experiences in the classroom, and that these beliefs "shape their behavior in ways that have extraordinarily powerful (and often negative) consequences" (p. 359).

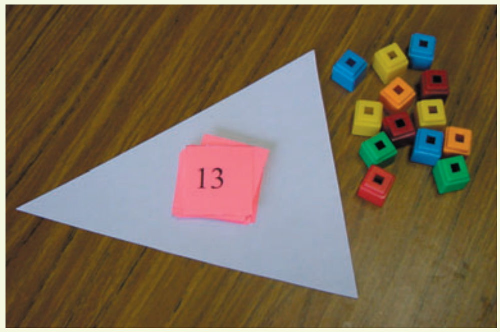
He cites his own research (Schoenfeld 1988) involving a survey of two hundred twenty-seven high school mathematics students in grades 9 to 12. When asked, "If you understand the materials, how long should it take to answer a typical homework problem?" the students gave answers that averaged 2.2 minutes. The same students were asked, "What is a reasonable amount of time to work on a problem before you know it's impossible?" and their responses averaged 11.7 minutes.

Although these were high school students, if classroom experiences largely shape these beliefs, patterns presumably begin to be established in the early years of elementary school. Most readers would be concerned about these students' views of the discipline of mathematics and what it means to do mathematics. It could be argued that teachers in all grade levels have a responsibility, by their words and actions, to present a different view.

So what can the typical classroom teacher do? Several authors (Bird 1999; Folkson 1995) have

Figure 1

The materials that students used



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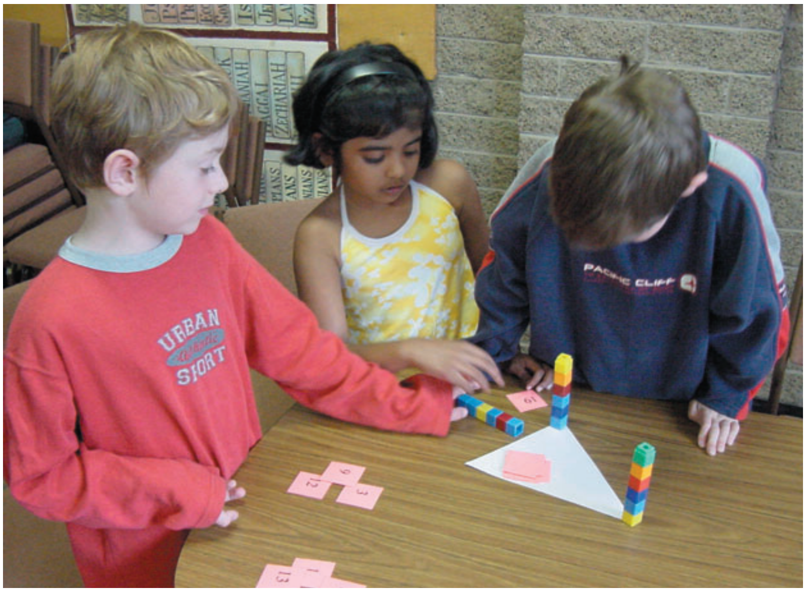
By Doug M. Clarke and Barbara A. Clarke

Doug Clarke, d.clarke@patrick.acu.edu.au, is director of the Mathematics Teaching and Learning Centre at Australian Catholic University in Melbourne, Australia. His professional interests include young children's number learning, teacher professional growth, and working in classrooms with teachers and children to make mathematics challenging and enjoyable. Barbara Clarke, Barbara.Clarke@education.monash.edu.au, teaches preservice and postgraduate students at Monash University in Victoria, Australia. Her interests include young children's spatial reasoning, teacher professional growth, and assessment.



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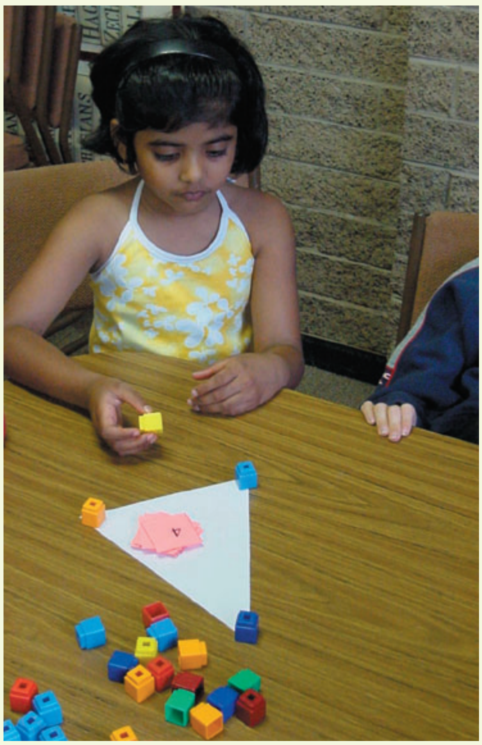
illustrated how the use of challenging and engaging problems can have an impact on children's beliefs about mathematics and themselves as learners.

One strategy is for the teacher to model the struggle that is often involved in solving genuine problems. The teacher can introduce a problem that he or she has not yet solved and show students the kind of process required to make progress on such a problem. Simply showing students that the teacher cannot solve a particular problem immediately will be most informative for many children who may have thought that teachers can solve and have solved every problem.

This article shares the experience of a class of second graders as they "wrestled" with two problems in a supportive yet challenging environment and saw the benefits of the struggle: achieving worthwhile solutions and considerable satisfaction from their efforts.

Figure 2

Placing the last cube



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Background

The classroom experiences that this article discusses took place as part of the Early Numeracy Research Project in Victoria, Australia. Three hundred fifty K–2 teachers in thirty-five schools participated in a three-year research and professional development project, exploring the most effective approaches to the teaching of mathematics in the first three years of school. The following were three essential components of this project:

- A research-based framework of "growth points" in young children's mathematical learning (in number, measurement, and geometry), highlighting typical learning trajectories and important stepping stones in children's thinking and strategies

- A forty-minute, one-on-one interview that all teachers used with all children at the beginning and end of the school year (at the time of this writing, the interview had been used with more than thirty-six thousand children in grades K–4)
- Extensive professional development at central, regional, and school levels, for all teachers, coordinators, and principals, with the focus on taking what was learned from the interview and day-to-day interactions with children to inform planning and teaching for maximum effectiveness, both cognitive and affective

For further information on the project, see Clarke (2001), Clarke et al. (2002, 2003), and Sullivan et al. (2000).

As part of the professional development for the project, the research team made five hundred seventy-eight visits to schools, working with teachers, children, principals, mathematics coordinators, and parents. The team worked in classrooms on each visit, either joining the regular mathematics activities of the class or team-teaching with the regular classroom teacher, often trying mathematics tasks and problems that the researchers or teachers had not used before. The following discussion concerns two such problems.

The First Problem

Ararat North Primary School is a small country school in a farming community in Victoria. During a school visit, one of the authors presented Anne Joyce's grade 2 class with an activity that led students to pose another problem. The activity was adapted from the *Primary Initiatives in Mathematics Education* materials from England (see Shuard 1992). The experiences described here took place in the latter part of the school year, when most grade 2 children are about eight years old.

Following a whole-group introduction, the researcher and teacher gave the children cardboard triangles, a pile of Unifix cubes, and small cards numbered 1 to 20 (see **fig. 1**). Working in groups of three, the children shuffled the cards. Then they turned over a card, took that number of Unifix cubes, and built three towers, one in each corner of the triangle. We told the children to make the three towers as close in height to one another as possible. For each number that the children turned over, they had to record in some way whether the towers could all be the same height. For example, twelve cubes would work but seven would not. As **figure 2** shows, sharing four would lead to unequal towers once the last cube is placed.

As the children worked, we moved around the room, encouraging them to share what they noticed. Some children had hypothesized that even numbers of cubes would lead to "even towers," an interesting example of how the everyday meaning of a word (*even*) can be somewhat different from its mathematical meaning.

After about thirty minutes, we brought the class back together again and groups shared what they had found. The table on the whiteboard summarized their findings (see **fig. 3**). We invited the children to make conjectures about any patterns that were evident. Only one child pointed out counterexamples to the theory that even numbers made even towers; no other patterns were suggested. In discussions with the teachers later, we noted that this was further evidence of the challenges that concepts related to multiplicative thinking present to young children; earlier evidence had emerged in the one-on-one interviews. We made a deliberate decision to not put the left-hand column of the table into numerical order at this stage. The temptation for the teacher at this stage is to attempt to bring closure to the activity by leading the children to the desired pattern. Anne Joyce resisted this temptation, however, and she encouraged the children to continue to think about possible patterns over the coming days: "Each time you walk past the whiteboard over the next couple of days, see if you can see anything interesting there, and let me know what you find."

Figure 3

Students' findings

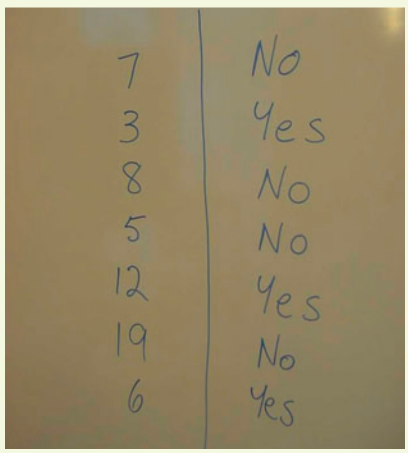
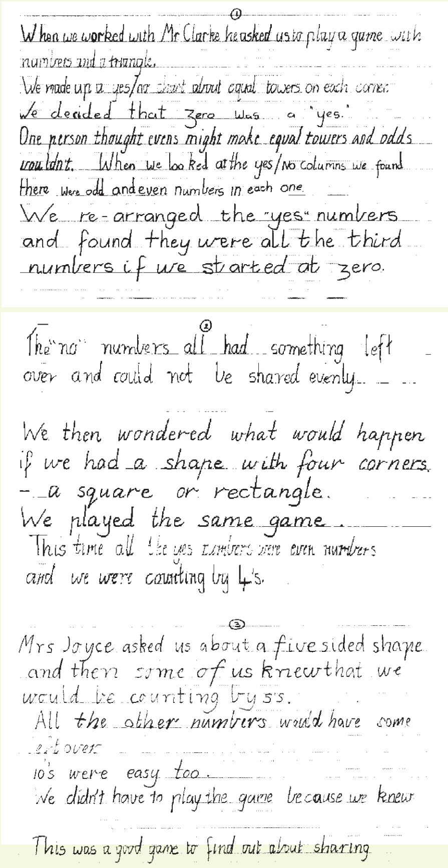


Figure 4

The children record their story



1

When we worked with Mr Clarke he asked us to play a game with numbers and a triangle.
We made up a yes/no chart about equal towers on each corner.
We decided that zero was a "yes."
One person thought evens might make equal towers and odds wouldn't. When we looked at the yes/No columns we found there were odd and even numbers in each one.
We re-arranged the "yes" numbers and found they were all the third numbers if we started at zero.

2

The "no" numbers all had something left over and could not be shared evenly.

We then wondered what would happen if we had a shape with four corners. – a square or rectangle.
We played the same game.
This time all the yes numbers were even numbers and we were counting by 4's.

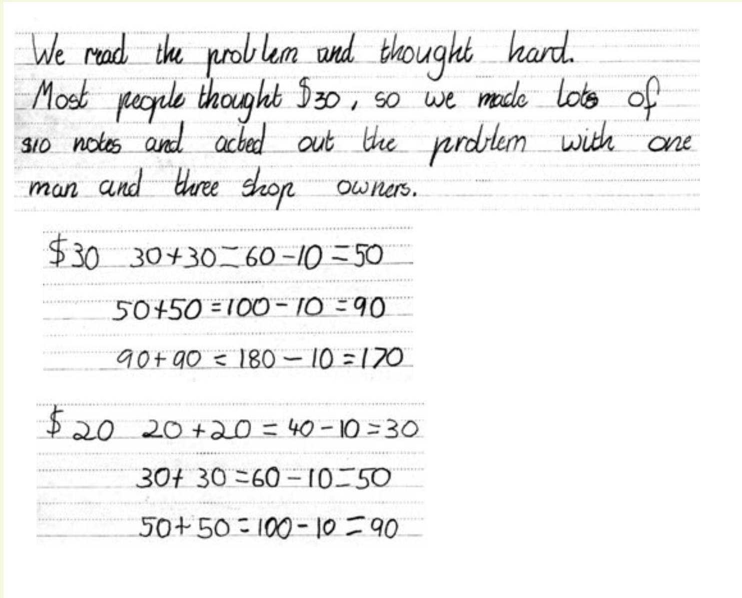
3

Mrs Joyce asked us about a five sided shape and then some of us knew that we would be counting by 5's.
All the other numbers would have some left over.
10's were easy too.
We didn't have to play the game because we knew

This was a good game to find out about sharing

Figure 5

Students' initial attempt at solving the problem



We read the problem and thought hard.
Most people thought $30, so we made lots of $10 notes and acted out the problem with one man and three shop owners.

$30 30+30=60−10=50
50+50=100−10=90
90+90=180−10=170

$20 20+20=40−10=30
30+30=60−10=50
50+50=100−10=90

A week later, all the project teachers met for a day of professional development. Anne Joyce had brought something for the authors from the children. She explained that the children had continued to consider the patterns in the findings and had kept a record; different children wrote parts of the story. She left this story with us (see **fig. 4**). We were excited not only about what the children had discovered but also about the way in which they had attempted to try the original problem for different shapes, leading to a level of generalization. They had persisted in working on what was for them a challenging problem.

The positive experience of working on this problem encouraged us to send the children another problem.

The Second Problem

Several years ago, we came across the following problem (source unknown):

> A man goes into a store and says to the owner, "Give me as much money as I have with me and I will spend $10." It is done, and the man does the same thing in a second and third store, after which he has no money left. How much did he start with?

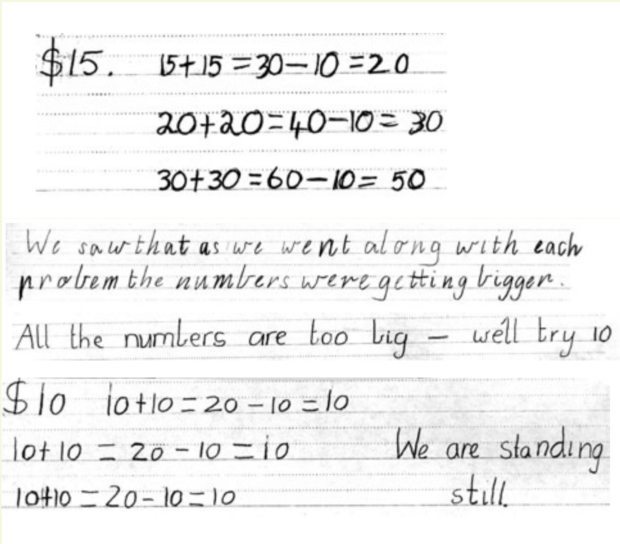
We invite readers to take some time to work through this problem. We have found it very useful over the years in our work with preservice teachers—in classroom sessions relating to problem-solving strategies. Of course, presenting the problem to preservice teachers is quite different from presenting it to eight-year-olds. Nevertheless, we faxed a copy of the problem to the children. We explained that it was extremely difficult but that we thought they might relish the challenge it provided.

We heard nothing from the school for about three weeks, until a large poster arrived late one Friday afternoon. The accompanying note from Anne Joyce explained that the students had continued to revisit the problem over a number of days. Once again, they had recorded their thinking and discoveries at each stage. Joyce had sent the students' work to us.

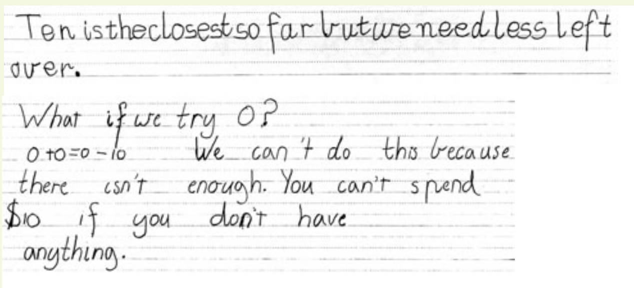
We were pleased to see that the children were encouraged to come up with an initial estimate, and we were interested to find that most students thought

Figure 6

The children tried out different values.


$15. 15+15 =30−10 =20
20+20=40−10= 30
30+30 =60−10= 50
We saw that as we went along with each problem the numbers were getting bigger.
All the numbers are too big — well try 10
$10 10+10=20−10=10
10+10 = 20 − 10 = 10 We are standing still
10+10 = 20 − 10 = 10

(a)


Ten is the closest so far but we need less left over.
What if we try 0?
0+0=0−10 We can't do this because there isn't enough. You can't spend $10 if you don't have anything.

(b)

the answer was $30, presumably because three times $10 is $30 (see **fig. 5**). Also interesting was that they had decided that acting out the story was a way of making sense of what the problem was actually asking. Physical involvement often is a powerful tool in the mathematics classroom. Of course, it was important to point out to the children that the use of the equals sign in their statements is incorrect and that both sides of these "equations" are not equivalent.

The children continued to try different values, but each time they ended up with a relatively high final figure (see **fig. 6a**). We liked the comment "We are standing still." At this stage, the children realized that they still needed to start with a lesser number (see **fig. 6b**). **Figure 7** shows a crucial point in the children's problem-solving process. The children had discovered what happens if you start with $5 and developed the strategy of working backward. Although the arithmetic was complicated, particularly for grade 2 children, they reached a solution after much effort (see **fig. 8**).

Some readers may be doubtful that typical second graders could do such work. Their teacher explained that after a while, the mathematics got too difficult for some children. Nevertheless, they all continued to participate in various ways, by taking notes, acting out the role of storekeeper, and so on. Children who could cope with the increasingly complex mathematics continued to pursue a solution.

The children concluded their summary with the following statement: "It took a lot of thinking and working out." We were thrilled with the quality of the students' thinking and persistence on such a difficult problem.

Grade 2 Children as Mathematical Thinkers

In an excellent discussion of mathematical thinking, Watson and Mason (1998) list nineteen "kinds of mental activity which, together, typify mathematical thinking" (p. 7). Of these, the following fourteen were evident in the children's work on these two problems: exemplifying, sorting, changing, reversing, generalizing, explaining, verifying, refuting, specializing, comparing, organizing, conjecturing, justifying, and convincing.

The children were engaged in mathematical thinking of an impressive kind, which gave great personal satisfaction to themselves and their teacher. This satisfaction, as well as the children's exposure to what it means to work mathematically, will be a positive influence on affective aspects of their learning as well.

Conclusion

This article began by discussing the worrisome beliefs about the nature of mathematics and doing mathematics that many students develop during their time in mathematics classes. Of particular concern is that students believe that if they cannot solve a problem almost immediately, it is impossible.

As this story illustrates, if teachers choose rich problems to use with children and encourage persistence, working together, making conjectures, sharing their findings, and allowing time for possibilities to emerge, a considerable chance exists that such beliefs will be less evident in the later years. We hope such beliefs will be replaced by a confidence that solving difficult mathematics problems that require ongoing struggle and persistence can give students considerable satisfaction and pleasure.

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Figure 7

A turning point in the students' problem-solving process

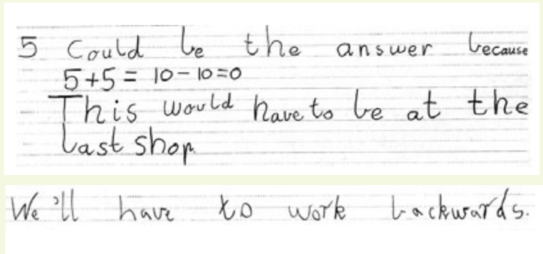

Figure 8

Students' solution