Iteration:

Unit Fraction Knowledge and the

French F

Using these tasks can help nurture children's multiplicative notions of unit fractions beyond part-whole understanding.

Between the Wholes:
Promoting Fraction
and Decimal Understanding

ry Tasks

Ron Tzur and Jessica Hunt

O ften, students who solve fraction tasks respond in ways that indicate inadequate conceptual grounding of unit fractions. Consider, for example, a student, Lia (all names are pseudonyms), who examined a long, rectangular piece of paper she had folded in the middle into two equal parts (halves).

"What fraction of the whole is one part?" asks Lia's teacher, Miss May.

Lia quickly answers, "One-half!"

Miss May is encouraged by Lia's response and says, "Now let's fold just this half into two parts. What fraction of the whole did we just make?"

The child hesitates, muttering, "One-third?" and looks down.

Puzzled, Miss May asks, "How do you know it's one-third?"

Lia explains, "Because it's one part out of three. On the fraction bar on the [classroom] wall, it goes, one-half, then a third, then a fourth. . . . "

Lia's erroneous response, we contend, not only is quite typical of many students but also reflects a conception rooted in prevalent practices of teaching unit fractions merely as "one out

of so-many equal parts of a given whole." Many elementary school curricula use folding, partitioning, shading, and naming parts of various wholes to develop children's understanding of unit and then nonunit fractions (e.g., coloring three of four parts of a pizza and naming it as three-fourths). Yet, using part-to-whole models for fractions rarely develops notions of rational numbers necessary for later proportional and algebraic reasoning (Post et al. 1992). In our collaborative research and as an alternative to the part-whole approach, we try to teach fractions and solidify children's multiplicative notions of unit fractions (Steffe and Olive 2010) through a core activity of unit iteration (i.e., using a single item, such as a paper strip of specific length, and repeating it a number of times to create and/or "measure" another unit). This

The authors explain how to set up the French Fry tasks and possible ways to use them (i.e., progression, representations, and teacher talk moves).

Task	Essential questions and notes
A: Share one fry equally between two people	**Goal:** Attach to the child's segmenting operations (observable through folding—child will fold initially. Folding will not be allowed as a strategy after task A). **Questions** • Tell me about your strategy. Why did you fold the paper into two parts? • What is the name of each part you created? How can you convince me that they are halves?
B: Share one fry equally among three people	**Goal:** Promote the child's iterating operations. Constrain the task such that child cannot fold or use a ruler to estimate the size of the share. **Questions** • I see your guess was [too long/too short]. Will your next try be longer or shorter? Why? • How much [longer/shorter] will you make your next guess? How do you know that is how much longer to make it?
C–D: Share one fry equally among four to five people	**Goal:** Promote the child's continued use of iterating operations. Practice the Repeat strategy (as opposed to segmenting, or double halving). **Within task questions** • I see your guess was [too long/too short]. Will your next try be longer or shorter? Why? • How much [longer/shorter] will you make your next guess? How do you know that is how much longer to make it? **Across task questions** • Always draw the child's attention to the size of the previous share before having him or her make a new share (reference the size of the share when sharing among three people before constructing the size of the share when sharing among four people). ○ Example: Before you make a guess about the size of the share among four people, look at the size of the share when we shared among three people. Will you make your next guess for the size of the share among four people longer or shorter? Why?
E–K: Share one fry equally among [6–12] people	• Have children play each other in pairs. Emphasize creation of share in the least number of attempts. ○ Continue with essential questions within and across tasks from above. ○ Begin to ask, "What is the size of this share called?" How do we write it?" [You may explain to children, "We call this one-ninth because the whole is nine times as large as each share or it takes exactly nine parts to remake the whole."]

article provides educators with an explanation of what we call the French Fry tasks—a series of tasks based on unit iteration that work to bring about children's unit fraction knowledge (Tzur 1999, 2007). Although the task is situated in the context of a French fry, which children and adults seemed to accept without trouble, other contexts that support the linear representation used in the tasks are also appropriate, should teachers find they make more sense in their classrooms (e.g., submarine sandwiches, sticks of clay, a beam of wood). When using iteration, the child's activity ensures equivalence of all parts a child produces by repeatedly copying an initial unit in contexts of equally sharing one (linear) whole among varying numbers of people. For example, when asked to share one whole French fry among five people, children estimate the size of the share that they believe, once iterated or repeated five times, will equal the size of the given whole. Children may be seen iterating a unit by, say, placing their fingers next to one another, moving them across the length of the fry. They may also move an eraser or another object (e.g., a piece of paper) they have deemed as the length of one person's share across the length of the whole. Iteration is a natural strategy to children, rooted in their existing conceptions in which units of one are repeated to conceive of whole numbers (e.g., iterating 1 six times, or 2 three times, produces the number 6). In a fractional sense, iteration helps children conceive of a whole as a multiple of the unit fraction, consisting of a certain number of copies of a same-size unit (Steffe and Olive 2010) that draws their attention to the number of times a unit fraction fits within the whole.

Through activities of iterating units, then, the child begins to understand unit fractions not just or mainly as shaded or folded pieces of a whole (e.g., one of five parts) but as a multiplicative relationship between a unit and the whole into which it fits a given number of times. In our example, the child comes to think of 1/5 as a unique quantity that, when repeated five times, exactly reproduces or fits inside of a referent whole. In fact, developing fraction knowledge through iteration provides children the conceptual basis to conceive of all rational numbers in this way (e.g., 4/5 is the iteration, or repetition, of 1/5 four times; $4 \times 1/5 = 4/5$) (see

Behr et al. 1992). The advantage of using unit iteration over, say, paper folding, is the possibility to generate any number of repetitions by adjusting the size of a single unit as opposed to the limited number of accurate folds a child (or any person) can possibly produce (e.g., halves, halves of halves, etc., and perhaps thirds, but not other numbers, such as 7 or 13).

In the following sections, we explain how to set up the French Fry tasks and possible ways to use them (i.e., progression, representations, and teacher talk moves—see **table 1**). We provide illustrations of tasks we have used in small-group settings and "snapshots" from the authors' research (Hunt, Tzur, and Westenskow, forthcoming) in small-group classroom settings that show children's thinking at various points in the task progression. These snapshots suggest formative measures for which a teacher can watch as children engage in solving the tasks. Finally, we provide solidifying activities for teachers to assess how children use their abstracted notions of unit fractions to solve story problems.

Task setup and critical features

The French Fry tasks begin with children being asked to share one whole fry among varying numbers of sharers. They begin with the use of concrete, tangible objects—namely, a yellow paper strip "French fry." As the tasks progress, the long, thin paper "fries" are eventually replaced with the use of computer software that is available as a free download (i.e., Javabars, http://math.coe.uga.edu/olive/welcome.html#LatestJBinstallers) (Biddlecomb, Olive, and Sutherland 2013). The software allows the creation of thin bars that simulate the fry as well as the actions (e.g., size estimation and/or adjustment iteration) that children use to manipulate the fry as they interact with the tasks. Teachers may want to familiarize themselves with the software before using the French Fry game with children (see the **online appendix** for an explanation of how to use the software).

Generally, we used the tasks for 20–30 minutes per day for a total of four to five days, a length of time teachers may need to vary based on children's evolving conceptions. In small-group settings, we posed the tasks to children in a think-pair-share way, where children first

For task A, students share a long, thin paper rectangle French fry equally between two people. In future tasks, the teacher constrains the task to solving without folding, promoting children's use of iteration.

Think: Children work to equally share the paper fry between two people.

Pair: Children discuss solutions with a partner.

Share: Children discuss strategies as a class.

Questions

- Tell me about your strategy.
- Why did you fold the paper into two parts?
- What is the name of each part your created?
- How can you convince me they are halves?

attempted the task on their own for a few minutes, then compared their solution methods with a partner, then shared and discussed solutions as a small group. Incidentally, in previous projects, we have also used the tasks in whole-class settings (Tzur 2007).

Fostering iteration

The initial task (A) involves children sharing a long, thin paper rectangle equally between two people (see **fig. 1**). Although many children will initially fold the paper strip to show the size of each person's share in task A, in future tasks, the teacher presents a constraint in the form of a challenge to solve the task without folding. This constraint promotes children's use of iteration to solve the tasks while extending the work on unit fractions beyond those limited to denominators that are multiples of two

("halving") or three ("thirding"). A teacher may gain an understanding of children's beginning notions of unit fractions by asking, for instance, "Why did you fold the paper into two parts to share?" and/or "What is the name for each part that you created? How can you convince me that these are halves?" In our work, children responded by suggesting that they can cut the pieces apart and verify the size of each piece as the same length. (Teachers might suggest this verification if children do not.) Children's justification of half would encompass the relative size of each of the parts to every other part and also to the whole; although, at this point, many children will merely conceive of halves as two equal "pieces." This means they are not yet paying attention to the size of the unit fraction with (multiplicative) respect to one whole.

Promoting and practicing iteration (via constraints)

In tasks B, C, and D, children consider a new, unmarked, and uncut fry and are asked to share it among three people, then move on to sharing among four and five people. Before students set off to share among three people, however, we present them with constraints: Do not fold the paper; do not use a ruler to measure the size of the share. We do this because we want to orient children to the use of iteration, or what we call the *repeat* strategy (Tzur 2000). In this strategy, the child (a) estimates the size of one person's share, (b) iterates that piece the number of times needed for people who share the entire French fry, (c) compares the iterated whole to the given one to be shared, and (d) continues from the first step by adjusting the size of a single share. When children place their fingers (or an object, such as an eraser or a piece of paper) next to one another, moving them across the length of the fry a number of times equal to the number of sharers, this indicates that the child is using iteration (see **fig. 2**).

At this point, a teacher may suggest that students use a piece of paper (other than the

French fry piece) to do the iteration, as opposed to fingers or another object. This allows children to accurately use the other piece of paper in iteration as well as to easily adjust the size of the iterated piece to better fit with the given whole (fry). During the pair portion of task B, a teacher may watch for and make public (i.e., name the iteration as the repeat strategy) children's strategies for determining the size of the share and then iterating (repeating) it as implied by the asked-for number of equal shares. If children use multiple strategies, such as trying to mark the yellow fry into thirds, or estimating the size of thirds from marking halves or fourths of the paper fry, a discussion might take place about which strategy is most effective and why (see **table 1** for essential discussion questions). For instance, students may point out that the repeat strategy is more precise than making visual estimates, or marks, on the paper fry; that it is easier to "test"; that it ensures all pieces are equal to one another; and that it can be used for virtually any number of sharers. In situations where the children do not initiate the strategy, a teacher might introduce and model it. For example, "I played this game with another group, and one child showed us this strategy [*show the use of the second paper as the share size, repeat across the yellow fry, and mark the repetitions*]. Do you think this strategy could work for us?"

As children use the repeat strategy, they take notice of the iterated part/share being either too long or too short. This noticing is based on their anticipation that the estimated share would be equal to the given French fry, whereas the actual, resulting, iterated whole may go over the length of the whole or not completely take up the length of the given whole (see **fig. 3**).

One way a teacher might help students keep track of their estimates is to provide an organizer (see **fig. 4**) as children practice the repeat strategy while sharing the French fry among four and five people (i.e., tasks D and E). Specifically, the teacher should ask essential questions as the children work: (a) "Will you make your
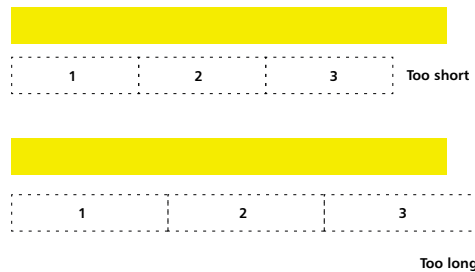
Placing fingers or an object side by side and moving them across the length of the fry a number of times equal to the number of sharers indicates the use of iteration.
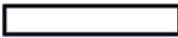
Based on their expectation that an estimated share would be equal to the given French fry, children using the repeat strategy will notice when the iterated share is too short or long.

next estimate longer or shorter than the previous ones? Why?" and (b) "How much longer or shorter? Why?"

These questions serve two important purposes. First, they help children begin anticipating the link between the nature of the adjustment to one (iterated) piece and the inverse order relation between unit fractions. This anticipation marks a conceptual change from the child's thinking when composing whole numbers by iterating same-size units of one. The composite whole number they create can become larger and larger with each iteration—that is, the size of the whole is not fixed. However, when children create unit fractions through the repeat strategy, both the unit whole and the unit fraction, or part, are fixed. The only thing that is left to vary is how big each part is: More iterations of a fixed part inside a fixed whole means smaller and smaller

An organizer can help students keep track of their estimates as they practice the repeat strategy while sharing a French fry among four and five people (tasks D and E).

One Whole French Fry	
Share among <u>four</u> people.	
Paste your estimate here.	Too long/too short? Why?
▭	Too short- didn't cover the whole thing.
▭	Too long- went over the whole.
▭	Too long- went over the whole.
▭	Got it! Made exactly the whole!

sizes of parts (and vice versa). As children make subsequent estimates regarding the size of shares, asking them if they would make the next one longer or shorter helps in promoting this intended anticipation. In a similar way, the teacher asks whether sharing among four people would yield a longer or shorter share than when they shared among three—and why. Once children have made their prediction, they construct their next piece and are asked what they discovered about their prediction.

Second, asking children to reflect on how much longer or shorter the next piece should be can eventually help them solidify their notions of the multiplicative attribute of the increase or decrease of one person's share. For example, when sharing a fry among six people, adding a small amount to an iterated piece would be replicated six times, not just one. In our work with children, they gradually began noticing this, which eventually turned into understanding the adjustment itself as a unit fraction of the overage/shortage. Most important, those adjustments and the child's focus on making

them precise leads to the complementing anticipation, namely, each unit fraction is unique in that it fits precisely the number of times the whole is to be shared. Said differently, for the child the unit fraction becomes determined by the number of times the whole is as much of the fraction. **Table 2** gives a progression of how we saw children's thinking change in terms of their understanding of the relative size of this adjustment.

Children realize over time that they have to adjust their next estimate relative to the number of iterations they use to create the whole (i.e., partition the leftover/shortage the number of times the share is being iterated). In this way, children finish inverting their notion of whole-number magnitude to that of unit fraction magnitude: The larger the number of iterations needed to produce one whole, the smaller the part/share should be. This serves as the conceptual basis for understanding why, if $n > m$, $1/m$ must necessarily be larger than $1/n$ (e.g., $1/6 > 1/5$ precisely because $6 > 5$). Children also come to understand the magnitude of any unit fraction as the result of partitioning a unit whole n times such that a remake of that whole comprises n iterations of a share size $1/n$.

We continued to promote the use of the repeat strategy as children made estimates of the size of the share in the remaining tasks (i.e., sharing the "fry" between four and five people), and we watched how children used the strategy to ensure that they were marking the repetitions of each estimate precisely. We also further discussed and made public (during solution sharing) how children were making decisions about the nature of the adjustment (longer or shorter) and the size of the adjustment (how *much* longer or shorter) of each estimate. Children discussed the iteration of estimates both within each task (e.g., "My previous estimate was too long because it went over the whole fry, so I had to make it shorter") and across tasks (e.g., when moving from fourths to fifths, "You have to fit more pieces inside the whole, so I had to make each share shorter"). We found in our work that children's reflection on these two essential questions helped them to abstract their notion of unit fractions as single magnitudes (even though written as two numbers—one above and one below the fraction bar). One way teachers might gauge children's evolving thinking is by
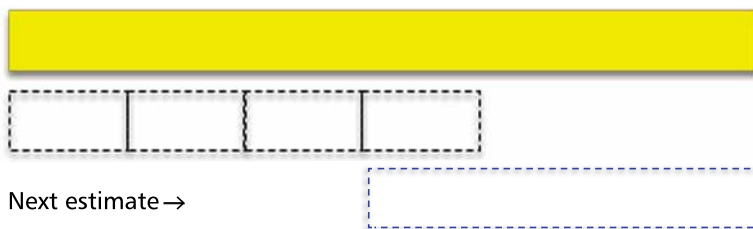
Progression of children's understanding of the relative size of an adjustment

|	
No anticipation of the nature of adjusting an estimate	Child either— Anticipates the nature of adjusting the estimate in whole numbers (he or she says the opposite of the nature of the adjustment needed), or Makes a random or seemingly wild guess of the nature of the adjustment needed. This may indicate that the child's anticipation of whole-number composite units needs to be developed.
Evolving anticipation of the nature of adjusting an estimate but not of its relative amount	Child anticipates "longer or shorter" with respect to the next estimate but has yet to anticipate "how much longer/shorter." A consideration of the relationship between the overage/shortage amount and the number of people sharing is absent. Child is likely to add/take off the entire shortage/overage amount to the next estimate. Next estimate →
Anticipation of the nature of adjusting an estimate with evolving relative amount	Child anticipates "longer or shorter" with respect to the next estimate and considers "how much longer or shorter" qualitatively. Child may say "a little bit." Child may use a guess-and-check strategy. Next estimate → "a little bit"
Anticipation of the nature and relative amount of adjustment	Child can tell "longer or shorter" with respect to the next estimate and determine "how much longer or shorter" by coordinating the amount of the overage/shortage with the number of people sharing (i.e., amount of adjustment is relative to the number of parts). Next estimate →
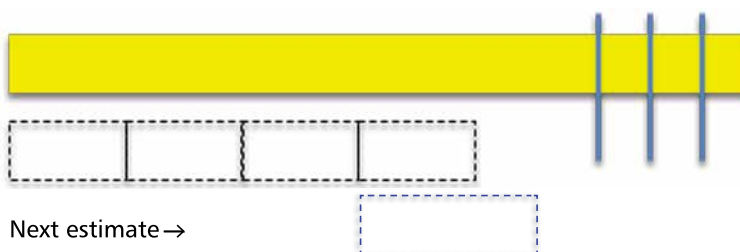
Iteration: Unit Fraction Knowledge and the French Fry Tasks

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms, and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs. The following questions related to "Iteration: Unit Fraction Knowledge and the French Fry Tasks" by Ron Tzur and Jessica Hunt are suggested prompts to aid you in reflecting on the article and on how the author's idea might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

1. What do the authors mean by "iteration"? How is determining a given fraction of a paper strip by iteration different than using paper folding? How do children go about using iteration to determine a specified fraction of a paper strip?

2. How is iteration related to multiplicative reasoning?

3. What benefits are there to encouraging adoption of the iteration or the "repeat strategy"? How does using this strategy help students to think about fraction concepts in productive ways?

4. How does each essential question presented in Figure 1 help to move students' thinking about fractions forward? What is the intention of each question?

5. What are different ways that children may interact with the tasks as their conceptions of unit fractions grow?

We invite you to tell us how you used Reflect and Discuss as part of your professional development. The TCM *Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to* Teaching Children Mathematics *at tcm@nctm.org. Please include Readers Exchange in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

using the levels of children's thinking outlined in **table 2** as a guide.

A playful task: Making precise shares in the fewest number of attempts

In the remaining tasks (i.e., constructing sixths through twelveths), children work to produce the size of each share (i.e., sharing one fry among six to twelve sharers) in as few estimates as possible. At this point, the tasks become somewhat game-like: Students continue to use iteration to make estimates about the size of each share or unit fraction as they work with a partner to see who can produce the share in the shortest number of attempts. The purpose of this added challenge is to further focus children's attention on the size adjustment of each share/unit as they work with situations involving more and more sharers. Teachers may want to use a game sheet (see **fig. 5**) to facilitate the game and record children's iterations. Students record the size of their estimates on the game sheet; each "round" is the number of people the fry is being shared with (the unit fraction being created). Only after children seem to understand both the uniqueness of the piece that fits *n* times into a given whole and the inverse relationship (i.e., Anticipation of Nature and Amount of Adjustment, see **fig. 4**), do we introduce formal fraction notation to the units/shares created. For instance, if round one involved sharing the fry among ten people, the unit fraction would be called *one-tenth* and notated as 1/10 while the teacher explicitly links this language to the number of times the share of one person must be repeated to fit within the whole.

Effective ways to use the French Fry tasks

Developing children's unit fraction knowledge is the foundation from which to build notions of all fractions as numbers/magnitudes. The French Fry tasks have been used in whole-group instruction as an alternative to existing curricula that center on the limiting part-of-whole activities to introduce fractions. The task sequence that we have presented could also be used alongside existing curricula in learning centers to enrich students' multiplicative understandings of fractions, or as an intervention for children who seem to need additional support in building fraction number sense. We found in our work that using iteration-based, equal-sharing activities is a natural, powerful, and enjoyable means of instruction for all children to develop and solidify their understanding of unit fraction quantities (Hunt, Tzur, and Westenskow, under review). We hope that you do, too.

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Children record the size of their estimates on a game sheet. Each round is the number of people the fry is being shared with (the unit fraction being created).

Round, task	Guess no.	Piece	Guess no.	Piece	Unit fraction created in relation to whole
One: Share the fry between or among _____ people.					
Winner of round one:_____ No. of guesses: _____ No. of points (equals no. of guesses: _____					
Two: Share the fry between or among _____ people					

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