KATM Bulletin

Kansas Association of Teachers of Mathematics

December 2016

2017 Fall Conference

Seaman Middle School in Topeka

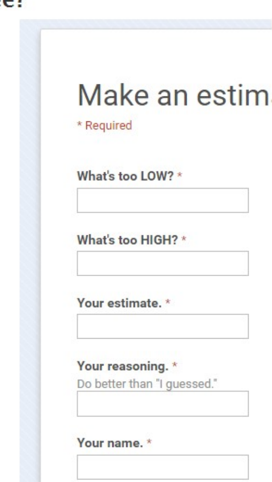
October 16, 2017

Keynote Speaker: Andrew Stadel

Andrew Stadel is involved in NCTM at the national level, who has given presentations across the country. He focuses on **Maximizing Effective Math Instruction by Crafting A Powerful Tool Belt**. Learn more about Mr. Stadel about at his blog. Mr. Stadel does a great deal of work with number sense and problem solving. He has tons of interesting estimation tasks through the Estimation 180 section of his website. How about this holiday related estimation task shown below to spark some discussion in your classroom?

How many lights are on the tree?

Additional information here.



Inside this issue:

A Message from our President

Hello Kansas mathematicians! First I want to thank everyone for their hard work to make the KATM 2016 conference a success. We had great conference with many presenters and vendors. A big thank you to Maize High School for hosting KATM and our fall conference.

In this issue of the KATM newsletter we are focusing on the standards of progressions. Specifically, we are looking at Number and Operations: Fractions to The Number Systems to Number and Quantity. Take a moment and read some of the great articles in our newsletter. We have some great articles from both our local officers and NCTM to share with our members. We have an interesting article on Engaging Students with Multiple Models of Fraction from the Teaching Children Mathematics magazine which can be found on page 8. We also have an engaging article about Mediants Make (Number) Sense of Fraction Foibles from the Mathematics Teacher magazine, found on page 29.

One of the goals of the KATM officer team is to provide engaging information and the latest news with mathematics in Kansas. We hope this newsletter offers some new ideas to try in your classroom and to share with your peers. NCTM has great resources available, and KATM is making an effort to provide some of the awesome resources from NCTM magazines to our KATM members. We are able to provide these great articles because KATM is an NCTM-affiliate organization….just one more way we can help Kansas math teachers.

As the new year is just around the corner, please mark your calendar for October 16, 2017. The 2017 Fall Conference will be held at the Seaman Middle School located in Topeka. We have already booked our keynote speaker, Andrew Stadel. We hope to see both new and familiar faces at this conference.

David C. Fernkopf

David C. Fernkopf

President, KATM

davidfernkopf@katm.org

Hello!

I hope that everyone's school year is going well. I don't know about you, but my year is going so fast…...I can't believe that December is here! Where has the year gone???? Hopefully this year will continue to go quickly and go well.

I hope that your December is productive as we start to eye second semester. Hopefully you will find this issue of the Bulletin full of useful resources.

Jenny Wilcox

Jenny Wilcox
KATM Bulletin Editor

CALL TO PRESENTERS:

If you're interested in presenting at the 2107 KATM Conference, we want to have a wide variety of sessions available. Feel free to contact jennywilcox@katm.org

Membership is up! As of October 13, 2016, there are 252 KATM members. We gained about 26 new members at the KATM Conference!

CALL FOR SUBMISSIONS

Your chance to publish and share your best ideas!

The KATM Bulletin needs submissions from K-12 teachers highlighting the mathematical practices listed above. Submissions could be any of the following:

◊ Lesson plans
◊ Classroom management tips
◊ Books reviews
◊ Classroom games
◊ Reviews of recently adopted resources
◊ Good problems for classroom use
◊

Email your submissions to our Bulletin editor: jennywilcox@katm.org

Acceptable formats for submissions: Microsoft Word document, Google doc, or PDF.

Common Core State Standards Mathematics Standards Progressions

Kindergarten	1	2	3	4	5	6	7	8	High School
Counting and Cardinality									Number and Quantity
Number and Operations in Base 10						Ratio and Proportional Relationships			
			Number and Operations: Fractions			The Number System			
Operations and Algebraic Thinking						Expressions and Equations			Algebra
								Functions	Functions
Geometry						Geometry			Geometry
Measurement and Data						Statistics and Probability			Statistic and Probability

Last year, KATM wrapped up our series on the mathematical practices. This year, we begin a new series, focused on the standards progressions. We will be focusing on how topics progress and change over the K-12 curriculum.

December 2016: Number and Operations: Fractions to The Number System to Number and Quantity

February 2017: Operations and Algebraic Thinking to Expressions and Equations/Functions to Algebra and Functions

April 2017: Geometry

October 2017: Measurement and Data to Statistics and Probability

Number and Operations in Base Ten to Ratios and Proportional Relationships to Number and Quantity

Number and Operation in Fractions (3-5)

3rd grade: Develop understanding of fractions (understand fractions as a whole partitioned into equal parts and a number on a number line); explain equivalence of special fractions and compare fractions

4th grade: Extend understanding of fraction equivalence and ordering, Build fractions from unit fractions (decompose fractions; add and subtract fractions with like denominators; multiply fractions by whole numbers). Understand decimal notation for fractions and compare decimal fractions.

5th grade: Understand equivalent fractions as a strategy to add and subtract fractions. Apply and extend previous understanding of multiplication and division. (interpret fractions as division; multiply fractions by fractions; interpret multiplication as scaling; divide unit fraction by whole numbers and vice versa)

Ratios and Proportional Relationships (6-8)

6th grade: Apply and extend previous understandings of multiplication and division to divide fractions by fractions (divide fractions and interpret quotients). Compute fluently with multi-digit numbers and find common factors and multiples. Apply and extend previous understandings of numbers to the system of rational numbers (understand positive and negative numbers and opposites; understand rational numbers as points on the number line; understand ordering and absolute value of rational numbers).

7th grade: Apply and extend previous understandings of operations with fractions (add, subtract, multiply and divide with rational numbers).

The Number System (High School)

The Real Number System: Extend the properties of exponents to rational exponents.

Use properties of rational and irrational numbers. **Quantities: Reason quantitatively and use units to solve problems.**

The Complex Number System: Perform arithmetic operations with complex numbers. Represent complex numbers and their operations on the complex plane. Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

Represent and model with vector quantities. Perform operations on vectors. Perform operations on matrices and use matrices in applications.

KSDE Update

Upcoming Events

KSDE Regional Trainings

Registration is now open for the KSDE Regional Math Trainings!

We have had 300 educators attend our first two trainings and have reported gaining loads of valuable information to use in their classrooms. Do miss out on this opportunity!

Please go to http://events.ksde.org/Default.aspx?tabid=852 for the agenda, session descriptions, and registration.

This will be a fantastic professional development opportunity for math educators and administrators in Kansas! Cost for the full day of training is only $35 per attendee.

These events will:

- Be full day trainings beginning at 8:00am and concluding by 4:00pm
- Have the same sessions so find the date that best fits your schedule
- Have sessions for administrators, elementary teachers/coaches, and MS/HS teachers/coaches

Dates and locations for the remaining trainings:

- February 6, 2017 – USD 259, Wichita
- February 27, 2017 – Greenbush Service Center, Girard

Topics that will be covered are:

- **Administrator:** Math look fors in the classroom, Family math night ideas, and Rethinking acceleration in middle/high school
- **Elementary:** Number sense, Common Situations (addition/subtraction & multiplication/division), and Fraction number sense
- **MS/HS:** Geometry and Functions

Kansas Excellence in Math and Science Education Conference

Save the date!

June 12th – 14th, 2017

Hutchinson, KS

Three full days of rich profession development in the areas of math and science.

Math Standards Review

Review process began in Spring 2016 and new standards are set to be sent to the State Board of Education for approval in June 2017. Detailed timeline, committee members and meeting dates can be found at http://community.ksde.org/Default.aspx?tabid=6151. KSDE will be holding town hall type meetings in January and February of 2017 and the standards will be up for public feedback during this time.

Math Science Partnership Grant

I am happy to announce that the Math and Science Partnership (MSP) Grant RFP and related materials have been released for this year. Application materials can be found at http://community.ksde.org/Default.aspx?tabid=5278.

AWARDS: Grant awards will range from $50,000 to $300,000 per year. In no instance will an annual award exceed $300,000. Grant activities may begin only after receipt of the grant approval notice. This is anticipated to be March 1, 2017. All grant activities must end by August 30, 2018.
DEADLINE: All applications must be delivered to Melissa Fast at the Kansas State Department of Education (KSDE) by **5:00 p.m. on Friday, December 9, 2016**. Faxed applications will not be accepted.
REQUIREMENTS: To be considered, KSDE must receive an original proposal, two copies, and one electronic copy by the date specified above. E-mail the electronic copy to Melissa Fast, mfast@ksde.org.

There will be an optional Bidders' Webinar on **November 18, 2016** which will provide an overview of the application process and serve to answer any questions from participants.
Training Opportunities
KSDE consultants and/or trained trainers can come to your district and provide training around many areas in mathematics. The cost to districts is very minimal and often time free of charge is a KSDE consultant can deliver the training. The training will be customized to the needs of the district. To request a training please go to http://community.ksde.org/Default.aspx?tabid=5812 and complete the training request form.

For questions related to mathematics in Kansas please contact Melissa Fast at mfast@ksde.org.

KATM
Kansas Association of Teachers of Mathematics

Working for ALL math teachers across Kansas

KATM Membership and our website

Are YOU a KATM member?

- If the answer is YES, thank you!
- If the answer is NO, becoming a member is a simple process. You can sign up at www.katm.org. The cost is very reasonable and there are several options.
- If you are unsure, feel free to contact Margie Hill at marghill@ku.edu or betsy.wiens@gmail.com. We would be glad to help you.

As a member of KATM you have access to a well-developed, informational KATM Bulletin, which is published four times a year. The bulletins can be found in our "members only" section of the KATM website. If you have never logged in to this website, you will need to reset your password. Please do not hesitate to contact Betsy Wiens if you need any support.

When checking and updating your membership information, PLEASE use a personal email if at all possible. We send membership renewal notices through our website and have discovered that-some district email servers will place emails arriving from an organization in a spam folder.

If your email, home address, job or other information has changed PLEASE update your information on the KATM website in the secure member area. Click on "My Account" to view your information and make edits.

Thank you for being a member of KATM!!!

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Engaging Students with Multiple Models of Fractions by

Xiaofen Zhang, M. A. (Ken) Clements, and Nerida F. Ellerton from *Teaching Children Mathematics*, October 2015

An understanding of unit fractions, and especially of one-half, one-third, and one-fourth, is crucially important for elementary school children's development of number sense (CCSSI 2010). We describe multimodal activities designed to assist elementary school students in gaining a rich understanding of unit fractions. Research has shown (Zhang 2012) that participation in the activities described in this article helped fi fth-grade students develop wider imagery and deepen their understandings with respect to unit fractions concepts. A body of literature is developing that is directed at teachers who wish to fruitfully engage elementary school students in concept-rich fractions activities (e.g., Ball 1993; Ball, Thames, and Phelps 2008; Barnett-Clarke et al. 2010; Chval et al. 2013).

A multiple-embodiment approach to teaching and learning fractions

The concept of a fraction is one of the most important mathematical ideas that elementary and middle school students encounter (Behr et al. 1984). However, the—

> difficulty with the learning of fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra. (U.S. DOE 2008, p. 28)

The view that an area-model approach should take priority in thinking about the teaching and learning of fractions has received much attention (Cramer and Henry 2002; NCTM 2000). Students are encouraged to represent fractions with plastic fraction pieces in "fraction kits" and draw or shade circular or polygonal (especially square and rectangular) representations of fractions.

Research indicates (Clements and Lean 1994; Samsiah 2002; Zhang 2012) that an overreliance on area-related contexts for teaching fraction concepts can result in students not coping well with fraction tasks that are presented in non area situations. Samsiah (2002) reported that the students in her research could answer questions based on the area model but could not transfer that knowledge to related, real-life situations. Zhang (2012) investigated the knowledge of fractions of fifth graders in the United States who had been taught the concept by methods in which area models had been almost exclusively emphasized and found that although the students could cope well with area-model tasks, many of them struggled when dealing with corresponding fraction tasks presented in non area-model contexts.

Some researchers have pointed to the importance of a multiple-embodiment approach to teaching and learning fractions so to assist students in deepening conceptual understandings of fractions (Moss and Case 1999; Zhang 2012). Moss and Case (1999) attributed the difficulties that many children experience when attempting to learn fractions to an overemphasis on area models; they advocated the use of "some alternative form of representation" (p. 144). Clarke, Roche, and Mitchell (2008) expressed regret that many students regard the area model as the only model of a fraction and proposed that if students are to understand fractions conceptually, they must become acquainted with a wider range of models.

From a mathematical perspective, the Common Core State Standards for Mathematics (CCSSM) has emphasized the importance of presenting fractions first and foremost as numbers with unique places on what will ultimately become known as "the real number line" (CCSSI 2010; Wu 2014). In the classroom, implementations of each of the six activities described below, the idea of placing fractions on a number line, was consciously and strongly emphasized. In any activity, identifying the meaning of the symbol m/n is important, where m and n are natural numbers and the result of a whole (or unit) having been partitioned into n equal parts, and m of those parts are being considered. The part-whole aspect of the concept definition is fundamental and must be emphasized in any activity aimed at developing sound fraction concepts.

Experiencing different representations of the same concept can help students abstract structural similarities and develop conceptual understandings for fractions. Students can be given opportunities to use different models when learning fractions. In what follows, we describe six multimodal activities that demonstrate how Develop a number line using paper strips to emphasize to students that the whole is the length between 0 and 1 on any of the three paper strips.

Three linear-model embodiments

Activity 1: Developing a number line using paper strips

Three narrow paper strips, each having the same length but different colors—let's say green, yellow, and purple—and a sheet of paper are prepared and made available to pairs of students. On the sheet of paper, a line segment, which is the same length as the three colored strips, is drawn, with the symbol "0" at one end and "1" at the other end (see fig. 1). This "zero-to-one" line segment is used to emphasize that the unit involved, the "whole," is the length between the positions for 0 and 1 on any one of the three paper strips.

FIGURE 1

Develop a number line using paper strips to emphasize to students that the whole is the length between 0 and 1 on any of the three paper strips.

Working in pairs, students fold the green paper strip into two equal parts, the yellow strip into three equal parts, and the purple strip into four equal parts. Then they unfold the pieces of paper and use them to locate positions for 1/2, 2/2, 1/3, 2/3, 3/3, 1/4, 2/4, 3/4, and 4/4 on the number line drawn on the sheet of paper. Before students attempt to locate the positions for the fractions on the number line, the teacher emphasizes that 2/3, for example, means "two lots of 1/3."

Activity 2: Folding a piece of rope

Working in pairs and using a piece of rope (about one yard long) and three different colored markers (red, blue, and green), students fold and then indicate the following: (a) exactly one-half of the piece of rope with a red mark on the rope; (b) exactly one-third and two-thirds of the piece of rope with blue marks; and (c) exactly one-fourth, two-fourths, and three-fourths of the piece of rope with green marks. Each pair of students then takes its piece of rope and checks whether the red, blue, and green marks correspond to the correct positions of 1/2, 1/3, 2/3, 1/4, 2/4, and 3/4 that have been placed on a number line drawn on the board (which has the symbol "0" at one end and "1" at the other). The distance from "0" to "1" is one yard; this constitutes the "whole." As with activity 1, the meaning of a fraction like 3/4 as "three lots of 1/4," is emphasized. Activity 2 is structurally and materially like activity 1 deliberately—both are likely to guide students toward the realization that a fraction is a number with a unique position on a number line. After students become familiar with activity 2, they can be invited to work on two in-class tasks (see fig. 2).

Activity 3: Developing a human number line

Thirteen students are called to the front of the classroom and are asked to space themselves evenly across the room, facing the class. The student on the left should hold a card showing “0” and should remain standing throughout the activity. The student on the right should hold a card showing “1.” The other students in the class are invited to be “expert observers.” The teacher draws attention to the fact that twelve spaces are between zero and one; through questioning, the class concludes that each space has a length of one-twelfth of the whole. Through further questioning, the class determines that successive students should be called one-twelfth, two-twelfths, three-twelfths, and so on.

FIGURE 2

The second activity, two fraction tasks using a piece of rope, is intentionally structured like the first activity to guide students toward understanding that a fraction has a unique position on a number line.

1. Find ***two*** ways to tell your partner, in words, how you could use the number line on the board and your piece of rope to answer the following:

   $2 \times \frac{1}{2} =$

   $3 \times \frac{1}{3} =$

2. Draw pictures and explain how you folded the rope for each of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

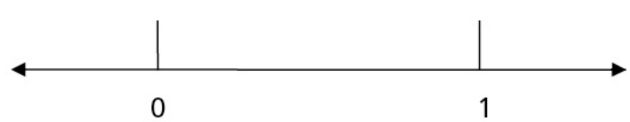
The activity continues with the twelve active participants (but not “0”) being asked to crouch down. Emphasize that the “whole” is the distance between the students representing zero and one. Then call out fraction names: one-half, one-twelfth, one-third, two-thirds, three-thirds, one-fourth, two-fourths, three-fourths four-fourths, five-twelfths, and so on. As each fraction name is called out, the “right” number of students—always starting at the zero end of the twelve students—stand up to represent the fraction. The response of standing, or not standing, helps students recognize that five-twelfths, for example, means that the distance between “0” and “5/12” is the sum of five separate distances of one-twelfth. It is also the total distance between “0” and “5/12.” Students can be invited to reflect on what “0/12” might mean.

This human number-line activity has the potential to facilitate mathematically rich thinking among young learners. Ideas of equivalent fractions can be discussed (“What is another name for nine-twelfths?”), and after a short time, the “expert observers” should be invited to suggest fraction names. The task can be varied so that, for example, nine students are at the front of the class (instead of thirteen students). In our experience, the activity assists students with recognizing that once the positions for zero and one are fixed, fractions have unique positions on a number line.

FIGURE 3

After students complete the first three activities, ask them to practice using their knowledge and skills with respect to fractions on a number line.

The number line below has space for some numbers less than 0 and some numbers more than 1. On this number line, show where $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ should be placed. Where should $\frac{2}{3}$ and $\frac{3}{4}$ be placed? Where should $\frac{1}{6}$ and $\frac{1}{8}$ be placed? Where should the fraction represented by $\frac{9}{8}$ be placed? Where should the fraction represented by $\frac{24}{12}$ be placed?



After completing activities 1, 2, and 3, ask students to discuss review questions or extensions (see fig. 3) so that they will practice, consolidate, and reflect on how to extend their knowledge and skills with respect to the number line.

A perimeter-model embodiment

Activity 4: Playing "fractions baseball"

Four students are invited to make the shape of a square, remembering to create four equal sides using a piece of rope about eight yards long. After doing this, they fix the square to the floor. Then an "imaginary game of baseball" can be played, beginning with two volunteers who are chosen to act as "pitcher" and "batter." The pitcher stands in the middle of the square and pretends to pitch to the batter, who stands at one corner (which corresponds to "0" and also to "1") and swings, pretending to try to hit the ball. Then the batter runs around the sides of the square a distance that should correspond to a fraction name (e.g., "one-fourth") called out by the teacher. Thus, if "one-fourth" were called, then the batter would run, in a nominated direction (usually counterclockwise, to correspond to the game of baseball) along one side of the square.

Then the next batter would take the "plate," and the exercise would be repeated, but three students would be involved (the pitcher and the first and second batters). During the game, calls of "one-fourth," "two-fourths," "three-fourths," and "four-fourths" can be made, as well as "foul" and "strike one," and so on. Students can also be asked to reflect on what the calls "one-half" and "one-third" might mean in the context of the game.

With activity 4, emphasize that the "whole" is the total distance around the square (or "diamond"). The square is like a number line depicting numbers from 0 to 1; the number line has been folded so that it has four equal sides. Also, three-fourths, for example, corresponds to running three of the four sides. Students can be invited to reflect on what one-half means and on why one-half is equal to two-fourths.

As an extension to the activity, use the same rope to create a boundary in the shape of an equilateral triangle and fix it on the floor. Students could play a similar game of Fractions Baseball. Present activities like those with the square (or diamond) baseball game and invite students to "run" one-third, two-thirds, or three-thirds of the way around the triangle. Ask students to reflect on what one-half, one-fourth, and one-sixth might indicate in this activity. Moreover, have them explain how they link called-out fraction names with the sides of the triangle.

To make the activity more interesting, form two teams of students, and develop a procedure for drawing "calls" that are written on cards and placed in a bag. Have teams keep score. Elementary school students enjoy this game and can learn a lot about fractions by playing it.

A capacity-model embodiment

Activity 5: Pouring water

For this activity, students work together in groups of three. Each group is allocated a full jug of water and four clear glasses (each glass having the same height and radius and having vertical rather than oblique or curved sides). Students pour a full glass of water and then, by pouring, share it as equally as they can among the three other glasses. They then pour the water back together and develop four stories involving pouring lemonade, individually or with their group, that correspond to the following:

(a) 1/3 + 1/3 + 1/3 = ___

(b) 1/3 + 1/3 = ___

(c) 1 – 1/3 = ____

(d) 1 – 1/3 – 1/3 = ____ (see fi g. 4)

Students conceived of the "whole" as one full glass of lemonade and linked this to the idea that a glass of lemonade would cost 25 cents. You might ask students to also work on other fraction tasks; for example, encourage them to make up, write, and illustrate a "sharing lemonade" story that demonstrates how one-third is more than one-fourth.

A discrete-model embodiment

Activity 6: Sharing precious diamonds

Students form pairs, and each pair receives twelve discrete identical blocks (described as precious diamonds). Emphasize that although there are twelve individual blocks, in fact, the "whole" comprises all twelve blocks. Pairs are asked to show and explain how the twelve diamonds can be shared equally among two, three, and then four friends. Experience has shown that the most common strategy used is to "deal" blocks, successively, to individual "friends;" but teachers should watch pairs to see which of them, if any, form and then partition rectangular arrays. If such partitioning does not occur intuitively, then suggest to pairs of students that they think about how such partitioning could be associated with fractions like one-half, one-third, one-fourth, and so on. Afterward, students can solve the two tasks (see fig. 5) individually and discuss with their partner how they determined their answers.

FIGURE 4

Three fifth-grade students created a group story illustrating 1 – 1/3 –1/3.

Work in groups of three. Talk with the others in your group, and work out the value of $1-\frac{1}{3}-\frac{1}{3}$ by pouring water. Then make up a "pouring lemonade" story and draw a picture about $1-\frac{1}{3}-\frac{1}{3}$.

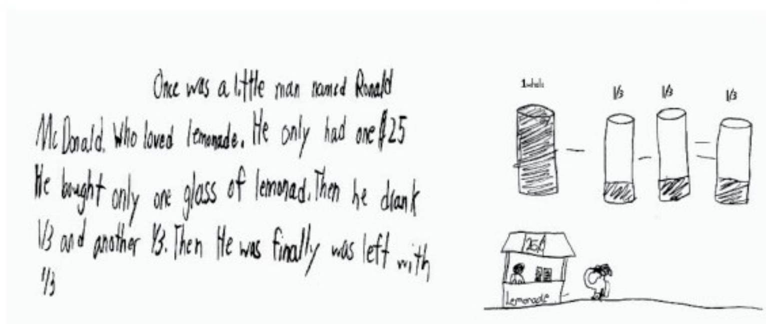
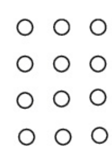


FIGURE 5

Before individuals solve the discrete model tasks, have student pairs partition and discuss rectangular arrays.

1. Suppose you arranged the 12 precious diamonds like this:



Draw a frame around the number of diamonds that would show $\frac{1}{2}$ of 12. Also, draw frames around the number of diamonds to show $\frac{1}{3}$ of 12 or $\frac{1}{4}$ of 12.

2. Draw arrays to show how to find the values of $\frac{1}{3}$ of 6, $\frac{1}{5}$ of 15, and $\frac{3}{4}$ of 20.

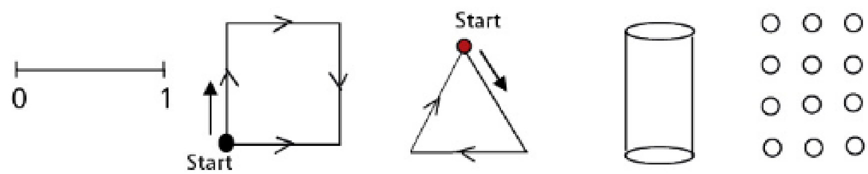
Reviewing

After the sequence of activities has been completed, distribute a review sheet to students (see fi g. 6). It might also be set up as a 6 × 6 grid, with the symbols 1/2, 1/3, 1/4, 2/3, 3/4, and 1 on the left side of the sheet and, to the right of these symbols, fi ve pictorial representations featuring distances on a number line, around a square, around an equilateral triangle, the amount of fl uid in a glass, and twelve identical objects. Ask students to indicate how one-fourth could be represented with the different models—linear, perimeter, capacity, and discrete models. As students complete the sheet and explain their solutions, select students to fi ll out the same table drawn on the board.

FIGURE 6

Below is a representation of one row of the review sheet used by the authors, five contexts to help students develop multiple representations of the fraction one-fourth.

Symbol	Fraction on a number line (linear model)	Fraction of the path around a square (perimeter model)	Fraction of the path around an equilateral triangle (perimeter model)	Fraction of a glass of water (capacity model)	Fraction based on 12 discrete objects (discrete model)
$\frac{1}{4}$					



General comments on the pedagogical approach

When a new activity is introduced to students, it is important to invite them to make connections between the new activity and those activities dealt with before, and to reflect on whether the various activities have something in common. Such an exercise can assist students in abstracting the symbols of fractions like one-half, one-third, and one-fourth out of the specific contexts and to build up a structural synthesis of conceptions of the associated fractions, thereby bolstering their understandings of fraction concepts.

Fifth graders who had engaged in these activities applied what they had learned in the Pouring Water and Playing Fractions Baseball activities to answer the requests in figure 2, which were used in the activity of folding a piece of rope. To illustrate 2 × 1/2 = 1, a student drew two half-full cups with one full cup. To illustrate 3 × 1/3 = 1, she drew three cups, each filled with one-third cup of liquid, and one full cup (see fi g. 7a). She could have improved her illustration if she had shown all the glasses as having the same height and volume. She used perimeters of an equilateral triangle and a square to find one-third and one-fourth of a piece of rope and drew details of how the triangle and square had been folded (see fi g. 7b).

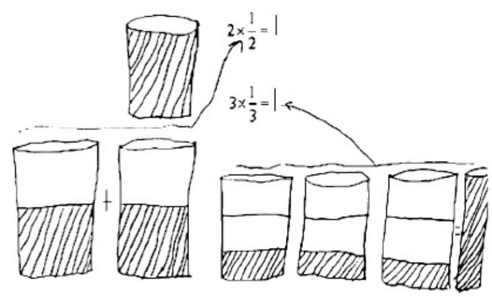
Promoting high-quality learning

An overemphasis on the area-model approach to teaching and learning fractions can hinder students' conceptual development of fractions. Research (e.g., Zhang 2012) suggests that a multiple-embodiment approach is more likely to promote high-quality student learning. The variety of models included in the six activities in this article can assist students in developing comprehensive concept images of fractions and facilitate their conceptual understandings of fractions. In Zhang's (2012) study, which featured a treatment and a control group from two fi fth-grade classes at the same school, students who participated in the activities developed much deeper conceptual understandings—as well as richer and coordinated concept images of one-fourth, one-third, one-half, two-thirds, and three-fourths—than students in the control group who had not participated in the activities. That said, the teachers who use these activities must emphasize what the "whole" is in each of the activities and how the same conceptual ideas are involved even though different embodiments of the concepts are presented.

FIGURE 7

This student could have improved her illustration if she had shown all the glasses as having the same height and volume.

(a) She drew two half-full cups with one full cup to illustrate 2 × 1/2 = 1. Three cups, each filled with one-third cup of liquid, and one full cup illustrate 3 × 1/3 = 1.



(b) She used the perimeters of an equilateral triangle and a square to find 1/3 and 1/4 of a piece of rope and showed how she had folded the triangle and square.

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Iteration: Unit Fraction Knowledge and the French Fry Tasks

by Ron Tzur and Jessica Hunt from Teaching Children Mathematics, October 2015

Often, students who solve fraction tasks respond in ways that indicate inadequate conceptual grounding of unit fractions. Consider, for example, a student, Lia (all names are pseudonyms), who examined a long, rectangular piece of paper she had folded in the middle into two equal parts (halves).

"What fraction of the whole is one part?" asks Lia's teacher, Miss May.

Lia quickly answers, "One-half!"

Miss May is encouraged by Lia's response and says, "Now let's fold just this half into two parts. What fraction of the whole did we just make?"

The child hesitates, muttering, "One-third?" and looks down.

Puzzled, Miss May asks, "How do you know it's one-third?"

Lia explains, "Because it's one part out of three. On the fraction bar on the [classroom] wall, it goes, one-half, then a third, then a fourth. . . . "

Lia's erroneous response, we contend, not only is quite typical of many students but also reflects a conception rooted in prevalent practices of teaching unit fractions merely as "one out of so-many equal parts of a given whole." Many elementary school curricula use folding, partitioning, shading, and naming parts of various wholes to develop children's understanding of unit and then non-unit fractions (e.g., coloring three of four parts of a pizza and naming it as three-fourths). Yet, using part-to-whole models for fractions rarely develops notions of rational numbers necessary for later proportional and algebraic reasoning (Post et al. 1992). In our collaborative research and as an alternative to the part-whole approach, we try to teach fractions and solidify children's multiplicative notions of unit fractions (Steffe and Olive 2010) through a core activity of unit iteration (i.e., using a single item, such as a paper strip of specific length, and repeating it a number of times to create and/or "measure" another unit). This article provides educators with an explanation of what we call the French Fry tasks—a series of tasks based on unit iteration that work to bring about children's unit fraction knowledge (Tzur 1999, 2007). Although the task is situated in the context of a French fry, which children and adults seemed to accept without trouble, other contexts that support the linear representation used in the tasks are also appropriate, should teachers find they make more sense in their classrooms (e.g., submarine sandwiches, sticks of clay, a beam of wood). When using iteration, the child's activity ensures equivalence of all parts a child produces by repeatedly copying an initial unit in contexts of equally sharing one (linear) whole among varying numbers of people. For example, when asked to share one whole French fry among five people, children estimate the size of the share that they believe, once iterated or repeated five times, will equal the size of the given whole. Children may be seen iterating a unit by, say, placing their fingers next to one another, moving them across the length of the fry. They may also move an eraser or another object (e.g., a piece of paper) they have deemed as the length of one person's share across the length of the whole. Iteration is a natural strategy to children, rooted in their existing conceptions in which units of one are repeated to conceive of whole numbers (e.g., iterating 1 six times, or 2 three times, produces the number 6). In a fractional sense, iteration helps children conceive of a whole as a multiple of the unit fraction, consisting of a certain number of copies of a same-size unit (Steffe and Olive 2010) that draws their attention to the number of times a unit fraction fits within the whole.

Through activities of iterating units, then, the child begins to understand unit fractions not just or mainly as shaded or folded pieces of a whole (e.g., one of five parts) but as a multiplicative relationship between a unit and the whole into which it fits a given number of times. In our example, the child comes to think of 1/5 as a unique quantity that, when repeated five times, exactly reproduces or fits inside of a referent whole. In fact, developing fraction knowledge through iteration provides children the conceptual basis to conceive of all rational numbers in this way (e.g., 4/5 is the iteration, or repetition, of 1/5 four times; $4 \times 1/5 = 4/5$) (see Behr et al. 1992). The advantage of using unit iteration over, say, paper folding, is the possibility to generate any number of repetitions by adjusting the size of a single unit as opposed to the limited number of accurate folds a child (or any person) can possibly produce (e.g., halves, halves of halves, etc., and perhaps thirds, but not other numbers, such as 7 or 13).

In the following sections, we explain how to set up the French Fry tasks and possible ways to use them (i.e., progression, representations, and teacher talk moves—see table 1). We provide illustrations of tasks we have used in small-group settings and "snapshots" from the authors' research (Hunt, Tzur, and Westenskow, forthcoming) in small-group classroom settings that show children's thinking at various points in the task progression. These snapshots suggest formative measures for which a teacher can watch as children engage in solving the tasks. Finally, we provide solidifying activities for teachers to assess how children use their abstracted notions of unit fractions to solve story problems.

Task setup and critical features

The French Fry tasks begin with children being asked to share one whole fry among varying numbers of sharers. They begin with the use of concrete, tangible objects—namely, a yellow paper strip "French fry." As the tasks progress, the long, thin paper "fries" are eventually replaced with the use of computer software that is available as a free download (i.e., Javabars, http://math.coe.uga.edu/olive/wel come.html#LatestJBinstallers) (Biddlecomb, Olive, and Sutherland 2013). The software allows the creation of thin bars that simulate the fry as well as the actions (e.g., size estimation and/or adjustment iteration) that children use to manipulate the fry as they interact with the tasks. Teachers may want to familiarize themselves with the software before using the French Fry game with children (see the online appendix for an explanation of how to use the software).

Generally, we used the tasks for 20–30 minutes per day for a total of four to five days, a length of time teachers may need to vary based on children's evolving conceptions. In small group settings, we posed the tasks to children in a think-pair-share way, where children first attempted the task on their own for a few minutes, then compared their solution methods with a partner, then shared and discussed solutions as a small group. Incidentally, in previous projects, we have also used the tasks in whole class settings (Tzur 2007).

Fostering iteration

The initial task (A) involves children sharing a long, thin paper rectangle equally between two people (see fig. 1). Although many children will initially fold the paper strip to show the size of each person's share in task A, in future tasks, the teacher presents a constraint in the form of a challenge to solve the task without folding. This constraint promotes children's use of iteration to solve the tasks while extending the work on unit fractions beyond those limited to denominators that are multiples of two ("halving") or three ("thirding"). A teacher may gain an understanding of children's beginning notions of unit fractions by asking, for instance, "Why did you fold the paper into two parts to share?" and/or "What is the name for each part that you created? How can you convince me that these are halves?" In our work, children responded by suggesting that they can cut the pieces apart and verify the size of each piece as the same length. (Teachers might suggest this verification if children do not.) Children's justification of half would encompass the relative size of each of the parts to every other part and also to the whole;

TABLE 1

The authors explain how to set up the French Fry tasks and possible ways to use them (i.e., progression, representations, and teacher talk moves).

Task		Essential questions and notes
A:	Share one fry equally between two people	**Goal:** Attach to the child's segmenting operations (observable through folding—child will fold initially. Folding will not be allowed as a strategy after task A). **Questions** • Tell me about your strategy. Why did you fold the paper into two parts? • What is the name of each part you created? How can you convince me that they are halves?
B:	Share one fry equally among three people	**Goal:** Promote the child's iterating operations. Constrain the task such that child cannot fold or use a ruler to estimate the size of the share. **Questions** • I see your guess was [too long/too short]. Will your next try be longer or shorter? Why? • How much [longer/shorter] will you make your next guess? How do you know that is how much longer to make it?
C–D:	Share one fry equally among four to five people	**Goal:** Promote the child's continued use of iterating operations. Practice the Repeat strategy (as opposed to segmenting, or double halving). **Within task questions** • I see your guess was [too long/too short]. Will your next try be longer or shorter? Why? • How much [longer/shorter] will you make your next guess? How do you know that is how much longer to make it? **Across task questions** • Always draw the child's attention to the size of the previous share before having him or her make a new share (reference the size of the share when sharing among three people before constructing the size of the share when sharing among four people). o Example: Before you make a guess about the size of the share among four people, look at the size of the share when we shared among three people. Will you make your next guess for the size of the share among four people longer or shorter? Why?

E–K:	Share one fry equally among [6–12] people	• Have children play each other in pairs. Emphasize creation of share in the least number of attempts.
		  o Continue with essential questions within and across tasks from above.
		  o Begin to ask, "What is the size of this share called?" How do we write it?" [You may explain to children, "We call this one-ninth because the whole is nine times as large as each share or it takes exactly nine parts to remake the whole."]

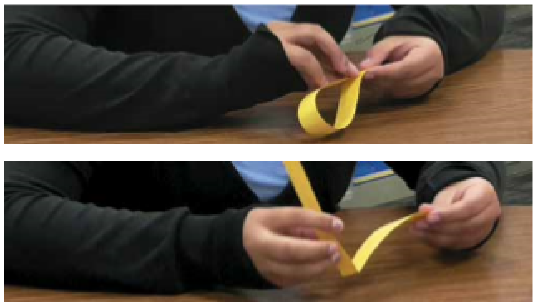
although, at this point, many children will merely conceive of halves as two equal "pieces." This means they are not yet paying attention to the size of the unit fraction with (multiplicative) respect to one whole.

Promoting and practicing iteration (via constraints)

In tasks B, C, and D, children consider a new, unmarked, and uncut fry and are asked to share it among three people, then move on to sharing among four and five people. Before students set off to share among three people, however, we present them with constraints: Do not fold the paper; do not use a ruler to measure the size of the share. We do this because we want to orient children to the use of iteration, or what we call the repeat strategy (Tzur 2000). In this strategy, the child (a) estimates the size of one person's share, (b) iterates that piece the number of times needed for people who share the entire French fry, (c) compares the iterated whole to the given one to be shared, and (d) continues from the first step by adjusting the size of a single share. When children place their fingers (or an object, such as an eraser or a piece of paper) next to one another, moving them across the length of the fry a number of times equal to the number of sharers, this indicates that the child is using iteration (see fig. 2).

FIGURE 1

For task A, students share a long, thin paper rectangle French fry equally between two people. In future tasks, the teacher constrains the task to solving without folding, promoting children's use of iteration.



Think: Children work to equally share the paper fry between two people.

Pair: Children discuss solutions with a partner.

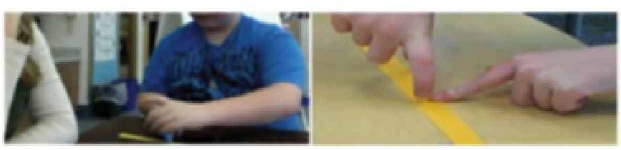
Share: Children discuss strategies as a class.

Questions

- Tell me about your strategy.
- Why did you fold the paper into two parts?
- What is the name of each part your created?
- How can you convince me they are halves?

FIGURE 2

Placing fingers or an object side by side and moving them across the length of the fry a number of times equal to the number of sharers indicates the use of iteration.



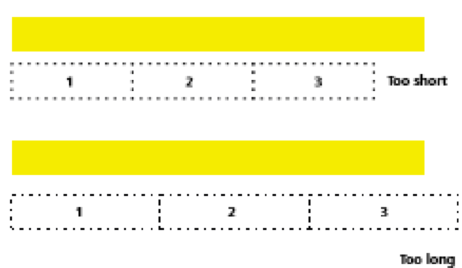
At this point, a teacher may suggest that students use a piece of paper (other than the French fry piece) to do the iteration, as opposed to fingers or another object. This allows children to accurately use the other piece of paper in iteration as well as to easily adjust the size of the iterated piece to better fit with the given whole (fry). During the pair portion of task B, a teacher may watch for and make public (i.e., name the iteration as the repeat strategy) children's strategies for determining the size of the share and then iterating (repeating) it as implied by the asked-for number of equal shares. If children use multiple strategies, such as trying to mark the yellow fry into thirds, or estimating the size of thirds from marking halves or fourths of the paper fry, a discussion might take place about which strategy is most effective and why (see table 1 for essential discussion questions). For instance, students may point out that the repeat strategy is more precise than making visual estimates, or marks, on the paper fry; that it is easier to "test"; that it ensures all pieces are equal to one another; and that it can be used for virtually any number of sharers. In situations where the children do not initiate the strategy, a teacher might introduce and model it.

For example, "I played this game with another group, and one child showed us this strategy [show the use of the second paper as the share size, repeat across the yellow fry, and mark the repetitions]. Do you think this strategy could work for us?"

As children use the repeat strategy, they take notice of the iterated part/share being either too long or too short. This noticing is based on their anticipation that the estimated share would be equal to the given French fry, whereas the actual, resulting, iterated whole may go over the length of the whole or not completely take up the length of the given whole (see fig. 3).

FIGURE 3

Based on their expectation that an estimated share would be equal to the given French fry, children using the repeat strategy will notice when the iterated share is too short or long.



One way a teacher might help students keep track of their estimates is to provide an organizer (see fig. 4) as children practice the repeat strategy while sharing the French fry among four and five people (i.e., tasks D and E). Specifically, the teacher should ask essential questions as the children work: (a) "Will you make your next estimate longer or shorter than the previous ones? Why?" and (b) "How much longer or shorter? Why?"

These questions serve two important purposes. First, they help children begin anticipating the link between the nature of the adjustment to one (iterated) piece and the inverse order relation between unit fractions. This anticipation marks a conceptual change from the child's thinking when composing whole numbers by iterating same-size units of one. The composite whole number they create can become larger and larger with each iteration—that is, the size of the whole is not fixed. However, when children create unit fractions through the repeat strategy, both the unit whole and the unit fraction, or part, are fixed. The only thing that is left to vary is how big each part is: More iterations of a fixed part inside a fixed whole means smaller and smaller sizes of parts (and vice versa). As children make subsequent estimates regarding the size of shares, asking them if they would make the next one longer or shorter helps in promoting this intended anticipation. In a similar way, the teacher asks whether sharing among four people would yield a longer or shorter share than when they shared among three—and why. Once children have made their prediction, they construct their next piece and are asked what they discovered about their prediction.

FIGURE 4

An organizer can help students keep track of their estimates as they practice the repeat strategy while sharing a French fry among four and five people (tasks D and E).

One Whole French Fry	
Share among four people.	
Paste your estimate here.	Too long/too short? Why?
	Too short- didn't cover the whole thing.
	Too long- went over the whole.
	Too long- went over the whole.
	Got it! Made exactly the whole!

Second, asking children to reflect on how much longer or shorter the next piece should be can eventually help them solidify their notions of the multiplicative attribute of the increase or decrease of one person's share. For example, when sharing a fry among six people, adding a small amount to an iterated piece would be replicated six times, not just one. In our work with children, they gradually began noticing this, which eventually turned into understanding the adjustment itself as a unit fraction of the overage/shortage. Most important, those adjustments and the child's focus on making them precise leads to the complementing anticipation, namely, each unit fraction is unique in that it fits precisely the number of times the whole is to be shared. Said differently, for the child the unit fraction becomes determined by the number of times the whole is as much of the fraction. Table 2 gives a progression of how we saw children's thinking change in terms of their understanding of the relative size of this adjustment.

Children realize over time that they have to adjust their next estimate relative to the number of iterations they use to create the whole (i.e., partition the leftover/shortage the number of times the share is being iterated). In this way, children finish inverting their notion of whole number magnitude to that of unit fraction magnitude: The larger the number of iterations needed to produce one whole, the smaller the part/share should be. This serves as the conceptual basis for understanding why, if $n > m$, $1/m$ must necessarily be larger than $1/n$ (e.g., $1/6 > 1/5$ precisely because $6 > 5$). Children also come to understand the magnitude of any unit fraction as the result of partitioning a unit whole n times such that a remake of that whole comprises n iterations of a share size $1/n$.

We continued to promote the use of the repeat strategy as children made estimates of the size of the share in the remaining tasks (i.e., sharing the "fry" between four and five people), and we watched how children used the strategy to ensure that they were marking the repetitions of each estimate precisely. We also further discussed and made public (during solution sharing) how children were making decisions about the nature of the adjustment (longer or shorter) and the size of the adjustment (how much longer or shorter) of each estimate. Children discussed the iteration of estimates both within each task (e.g., "My previous estimate was too long because it went over the whole fry, so I had to make it shorter") and across tasks (e.g., when moving from fourths to fifths, "You have to fit more pieces inside the whole, so I had to make each share shorter"). We found in our work that children's reflection on these two essential questions helped them to abstract their notion of unit fractions as single magnitudes (even though written as two numbers—one above and one below the fraction bar). One way teachers might gauge children's evolving thinking is by using the levels of children's thinking outlined in table 2 as a guide.

A playful task: Making precise shares in the fewest number of attempts

In the remaining tasks (i.e., constructing sixths through twelfths), children work to produce the size of each share (i.e., sharing one fry among six to twelve sharers) in as few estimates as possible. At this point, the tasks become somewhat game-like: Students continue to use iteration to make estimates about the size of each share or unit fraction as they work with a partner to see who can produce the share in the shortest number of attempts. The purpose of this added challenge is to further focus children's attention on the size adjustment of each share/unit as they work with situations involving more and more sharers. Teachers may want to use a game sheet (see fig. 5) to facilitate the game and record children's iterations. Students record the size of their estimates on the game sheet; each "round" is the number of people the fry is being shared with (the unit fraction being created). Only after children seem to understand both the uniqueness of the piece that fits n times into a given whole and the inverse relationship (i.e., Anticipation of Nature and Amount of Adjustment, see fig. 4), do we introduce formal fraction notation to the units/shares created. For instance, if round one involved sharing the fry among ten people, the unit fraction would be called one-tenth and notated as 1/10 while the teacher explicitly links this language to the number of times the share of one person must be repeated to fit within the whole.

Effective ways to use the French Fry tasks

Developing children's unit fraction knowledge is the foundation from which to build notions of all fractions as numbers/magnitudes. The French Fry tasks have been used in whole-group instruction as an alternative to existing curricula that center on the limiting part-of-whole activities to introduce fractions. The task sequence that we have presented could also be used alongside existing curricula in learning centers to enrich students' multiplicative understandings of fractions, or as an intervention for children who seem to need additional support in building fraction number sense. We found in our work that using iteration-based, equal-sharing activities is a natural, powerful, and enjoyable means of instruction for all children to develop and solidify their understanding of unit fraction quantities (Hunt, Tzur, and Westenskow, under review). We hope that you do, too.

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TABLE 2

Progression of children's understanding of the relative size of an adjustment

No anticipation of the nature of adjusting an estimate

Child either—

Anticipates the nature of adjusting the estimate in whole numbers (he or she says the opposite of the nature of the adjustment needed), or

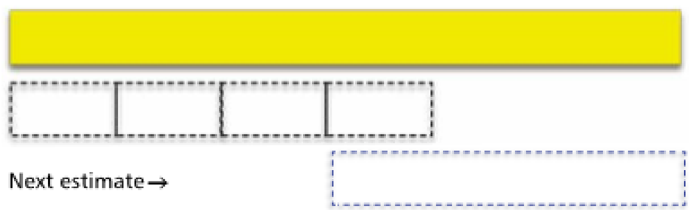
Makes a random or seemingly wild guess of the nature of the adjustment needed. This may indicate that the child's anticipation of whole-number composite units needs to be developed.

Evolving anticipation of the nature of adjusting an estimate but not of its relative amount

Child anticipates "longer or shorter" with respect to the next estimate but has yet to anticipate "how much longer/shorter."

A consideration of the relationship between the overage/shortage amount and the number of people sharing is absent.

Child is likely to add/take off the entire shortage/overage amount to the next estimate.

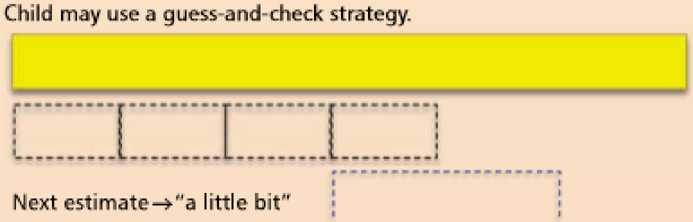


Anticipation of the nature of adjusting an estimate with evolving relative amount

Child anticipates "longer or shorter" with respect to the next estimate and considers "how much longer or shorter" qualitatively.

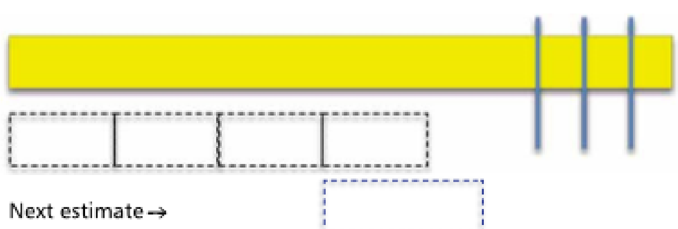
Child may say "a little bit."

Child may use a guess-and-check strategy.



Anticipation of the nature and relative amount of adjustment

Child can tell "longer or shorter" with respect to the next estimate and determine "how much longer or shorter" by coordinating the amount of the overage/shortage with the number of people sharing (i.e., amount of adjustment is relative to the number of parts).



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Amy Johnston, KATM VP Elementary

"That's not fair!" It is hard for me to believe there isn't a teacher or parent of children that doesn't hear this statement more than they care to admit. Kids are very sensitive to when things are unfair, making teaching the basics to "equal parts" a pretty easy task for a primary teacher. I can draw any shape, tell a story about it being a dessert of some sort or pizza, and after a little partner talk students are able to tell me if the shape is or is not divided into equal parts with impressive accuracy. Without even thinking twice teachers are making use of Mathematical Practice 6: Attend to Precision by asking students to explain to their partner or the class what is meant by equal parts.

Once students have a solid foundation of some basic vocabulary, such as whole and equal parts, we move them into reading different fractions and understanding how the fractions they are saying and writing correspond to their number line and they are just parts of whole numbers. As we have all seen, for students that struggle to have a solid understanding of numbers that this will be difficult. My suggestion for these students is to pull out manipulatives and let them work and explain fractions with concrete tools.

As students move into more difficult fraction lessons such as equivalent fractions, adding and subtracting fractions, and multiplying or dividing fractions, be sure to let them talk about what they are doing so we make sure students are not just learning how to do the math but missing the understanding of what they are doing. If you are interested in what others are saying about how they are teaching fractions in their classrooms, check out some of the links below:

Number Talk Fractions by Kristin Gray

Let's Talk Fractions by Desiree

Fun with Fractions by We Are Teachers

How to Teach Fractions by Jason Gibson

Do Twelfths Terminate or Repeat?

by Rebecca Ambrose and Erica Burnison from Mathematics Teaching in the Middle School, November 2015

When we find the decimal equivalent of a fraction with 12 in the denominator, will it terminate or repeat? This question came from a seventh grader in author Erica Burnison's class as the student was pondering a poster generated by one of her classmates. Not only did we find the question intriguing, but it also affirmed our belief in the power of tailoring instruction to students' interests and needs.

As math educators, trying to figure out how to implement the content and Standards for Mathematical Practice (SMP) found in the Common Core State Standards for Mathematics in a student-centered fashion, we were thrilled by this moment. This question from a student also demonstrated that students can engage in the SMP while working in collaborative groups.

In this article, we will describe the activities that gave rise to this student-generated question. We will also provide our collaborative reflection of the activities because we find that collaboration deepens our understanding of pedagogy as well as mathematics. We hope that our account will inspire others to work together.

We work in a university-school partnership, Innovations in STEM Teaching, Achievement, and Research (I-STAR). Burnison is also involved with the BetterLesson Master Teacher Project, where she documented her teaching in 2013–14. These two projects gave her access to video recordings of her students' conversations. Viewing videotapes of group discussions after class enabled her to eavesdrop on her students and use their ideas to tailor subsequent lessons to address their needs as well as meet the content standards.

INITIAL LESSON SET UP

The focus of the lessons we describe explores a grade 7 number standard in the Common Core: "Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats" (CCSSI 2010, 7.NS 2d; p. 49). Because Burnison's seventh grade students typically exhibited a negative attitude about their math abilities and tended to be fearful of anything related to rational numbers, she anticipated knowledge gaps. She wanted to challenge her students' misconceptions, so she began by asking them to convert the fraction 5/8 to a decimal. When they asked, "Does the 5 or the 8 go in the house?" and did not know whether 0.625 or 1.6 was a reasonable answer, she knew that they needed to develop better rational number sense. Difficulties with decimals are typical for many students, as identified in the research literature (Moloney and Stacey 1997). Burnison hoped to change the students' mind-set that fractions and decimals were the enemy by providing them with an opportunity to explore rational numbers in a nonthreatening way.

Armed with the information about students' misconceptions, she created a set of fractions for small cooperative groups to use in a sorting task. She strategically chose fractions in the hopes that students would observe patterns that might spark conjectures and questions (see fig. 1). She wanted to hear her students talk about "What if" and "Why" rather than "How to" as they developed a better understanding of the connection between fractions and decimals. She also hoped the task would help them confront some of their misconceptions.

To complete the task, students converted each fraction into a decimal and then used a calculator to doublecheck their work. Once they had the equivalent fraction and decimal pairs, students were asked to sort the fractions into sets in any way that made sense to them. Burnison did not care how they chose to sort them; she wanted them to look beyond the procedures that seemed to defeat them and instead experience the excitement of discovery that she hoped would emerge from noticing patterns.

Students represented their sorting discoveries on posters (see fig. 2). Some groups sorted the fractions according to same denominators; others separated repeating and terminating decimals; and others distinguished between fractions that were more or less than one whole. The variety of responses provided fodder for good discussions in future lessons, so the effort to tailor instruction to student needs could continue. At this point in the unit, students had experienced using division to convert fractions to decimals. Their engagement with the SMP had only just begun.

ONE GROUP'S INTERACTION

Burnison examined the posters and decided that she would provide prompts to engage students in examining one another's work and thinking about what classmates had completed. The prompts helped students use the organizational patterns created by their classmates to explore the Common Core's content standard.

The prompt that went with the poster in figure 2 was this:

> Is there anything you notice about the fractions that tells you in which place value the decimal ends or how the decimal ends?

These students had grouped all the fractions with the same denominator. In viewing the arrangements, Sam's, Elena's, and Juana's attention was drawn to the fractions with 8 in the denominator. They claimed that "when you divide by 8, it always ends in the thousandths place and it always ends in a 5." At this point, the trio had already noticed that fractions with 8 in the denominator convert to decimals that terminate with a 5 in the thousandths place. Sam continued to look for additional examples to support their claim, but, finding none, moved on to the next pattern.

Fig. 1 Students were asked to produce equivalent fraction and decimal pairs from the set below, then sort them in any way they pleased.

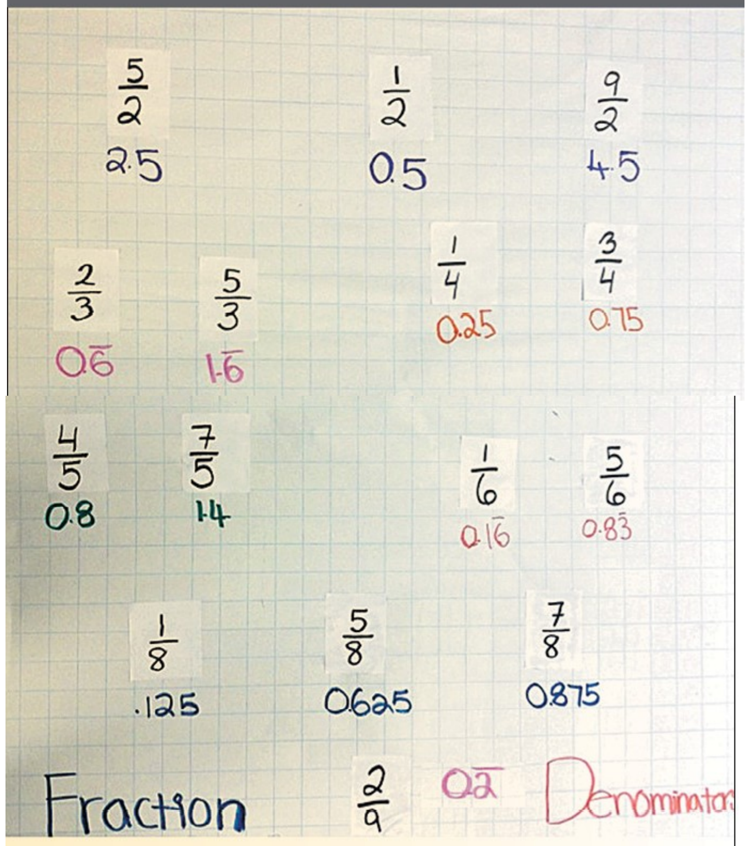
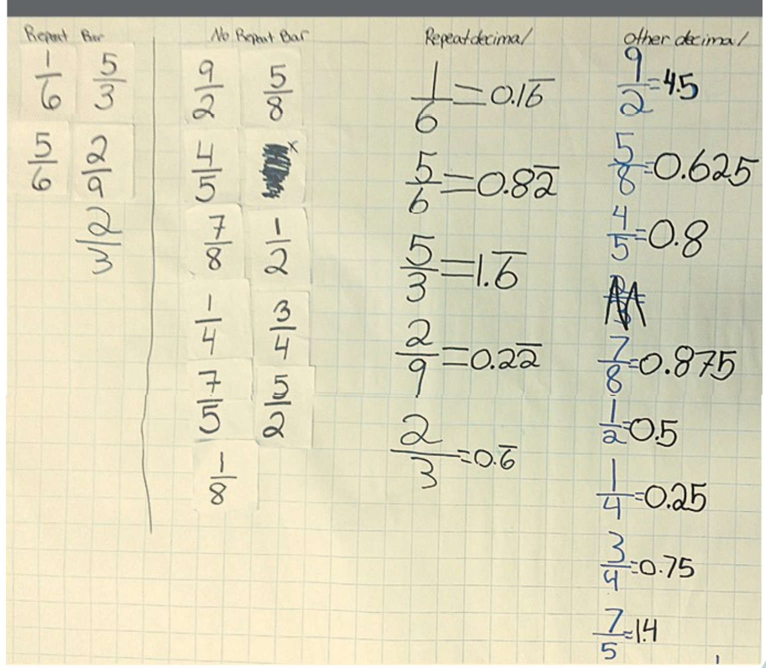


Fig. 2 It was hoped that students would find patterns while they were sorting fractions.



In retrospect, we realized that an opportunity was missed to suggest to the students that they generate their own data by providing more examples (i.e., 2/8 or 3/8) and testing their claim; however, we were happy that at least Sam realized that more data would strengthen his argument.

Elena then asked the group, "What about dividing by 2, 4, or 6?" They scanned the poster again and noticed that dividing by 2 also results in a decimal that "ends in a 5." Elena then pointed to 5/2 = 2.5, 9/2 = 4.5, and 1/2 = 0.5 to illustrate her point. Sam repeated the conjecture that "when you divide by 2, when the denominator is 2, it ends in a 5." These students had recognized another pattern in the relationship between the denominator and the terminating decimal equivalent, but had not asked why it occurred. More observations ensued.

Elena: Well, dividing by 2 and 4, it will end in a 5, too.

Sam: [Continued focusing on fractions with 2 in the denominator] These are all tenths. [Then he pointed to 3/4 = 0.75.] But this one ends in the hundredths, and the eighths ends in the thousandths. They all have 5.

Juana: Not all of them.

Sam: [Revising his statement] Yeah, the higher the number it is, the higher the denominator; well, it has to be 2, 4, 8.

Elena: Wait, are you saying all of them? All of them end in 5?

Sam: Only 2, 4, and 8 all end in 5, but 8s always in the. . .

Juana: Thousandths.

Sam and Juana then stated the rest of their claim together as they pointed to examples on the poster, "Twos always in the tenths, and fourths are in the hundredths."

Elena was particularly persistent in her critique by asking which fractions Sam was talking about, causing Sam to engage in SMP 6, "Attend to precision," as he specified which fractions ended in a 5.

ANOTHER GROUP'S INTERACTION

Another group had created a poster that separated the fractions into categories of repeating and nonrepeating decimals. Burnison decided to use this poster, illustrated in figure 3, to help students explore the patterns in repeating decimals and the division that produced them.

This group noticed that all the denominators for the repeating decimals were multiples of 3 (i.e., 3, 6, and 9). When Adam listed the denominators out loud, he did not stop at the examples provided on the poster but extended his pattern to denominators of 12. Isabella immediately questioned his assertion and pointed out that 12 is a multiple of both 3 and 4. She wondered if, when converting fractions to decimals, a denominator of 12 would act more like 3 and repeat or more like 4 and terminate ending in 5. Burnison was delighted that Adam extended the pattern and that Isabella critiqued his reasoning.

Both groups of students had noticed and expressed regularity (SMP 8) and had identified and used the structure of the number system (SMP 7). The first group noticed two regularities: The fractions whose equivalent decimal ended in 5 had denominators of 2, 4, and 8 and, within that group of fractions, the number of digits after the decimal point increased by 1 as the denominator doubled. The second group noticed that all the repeating had fraction equivalents with a multiple of 3 in the denominator. Moreover, they were extending the pattern to fractions with a denominator of 12 and conjecturing about how its decimal equivalents would behave. By articulating their observations to one another, they were working toward SMP 3, "Construct a viable argument and critique the reasoning of others." Although their arguments could use some fine-tuning, we were excited to see them engaging independently in the Common Core's Standards for Mathematical Practice. They understood Burnison's expectation that they make sense of things for themselves and share their ideas with classmates, and their interactions indicated that they were wholeheartedly participating in the activity. Their discussions provided Burnison with a window into their thinking and another opportunity to tailor her instruction to their needs.

STUDENTS' QUESTIONS

The next day, the class was told about the question that Adam and Isabella had brought up, "What happens when fractions with 12 in the denominator are turned into decimals?" and were invited to figure it out for themselves. Burnison designed an activity sheet to help gather data, listing all the twelfths from 1/12 through 12/12. The students were eager to get started trying to answer their classmates' question. After simplifying the fractions, most students used the division algorithm to convert the fractions to decimals. Burnison was pleased to see that students were using their knowledge of the size of the fraction to decide how to set up the division, a procedure that initially had thwarted them.

In one group, Marianna suggested that all the fractions that could be simplified would be converted to repeating decimals. However, Francisco and Becka pointed to the fractions 3/12, 6/12, and 8/12 as counterexamples to Marianna's claim. The group persevered and continued to look for patterns in the data. Like most of the groups, these students concluded that the fractions, once written in lowest terms, did hold the key to deciding what denominators of 12 would do. Those that simplified to a denominator of 3 would convert to repeating decimals, and those that simplified to denominators of 2 or 4 would terminate with a 5 in the tenths and hundredths place, respectively.

Another group made a slightly different observation, asserting that if a factor of 4 was removed when simplifying, the resulting decimal would repeat. Alternatively, if a factor of 3 was removed, the fraction would convert to a terminating decimal. Students were discovering relationships among fractions, decimals, and division and engaging in the Standards for Mathematical Practice.

Burnison was so focused on accomplishing this goal that she missed an opportunity to dig deeper into the math when Miguel shared his thinking. While completing the activity sheet, Miguel realized that he did not need to use long division to convert fractions to decimals, but instead could think about equivalent fractions with powers of 10 in the denominator. He did not articulate this pattern, but noted that 3/4 = 75/100 = 0.75. In this case, Miguel had found a useful structure in the numbers (SMP 7). He stepped back and shifted perspective to notice the equivalent fractions embedded in the process of converting fractions to decimals. Burnison did not immediately appreciate this underlying big idea in the midst of instruction.

Fig. 3 This listing demonstrated why denominators with powers of 2 convert to decimals that end in a 5.

$$\frac{1}{2} = \frac{1}{2} \bullet \frac{5}{5} = \frac{5}{10} = 0.5$$

$$\frac{1}{4} = \frac{1}{2^2} = \frac{1}{2^2} \bullet \frac{5^2}{5^2} = \frac{5^2}{10^2} = \frac{25}{100} = 0.25$$

$$\frac{1}{8} = \frac{1}{2^3} = \frac{1}{2^3} \bullet \frac{5^3}{5^3} = \frac{5^3}{10^3} = \frac{125}{1000} = 0.125$$

Sometimes it is difficult to recognize these opportunities during class and make adjustments in the moment. After we, the authors, collaborated, the important mathematical idea that Miguel was approaching became clear. We realized that the students were looking for patterns but were not asking what was causing them. In particular, they did not provide a justification for why fractions with a power of 2 in the denominator (2, 4, 8) would terminate in a 5 when converted to a decimal.

Although we discussed the math in figure 3, we are not sure if seventh graders could have generated this analysis using exponents. Even so, Burnison appreciated the discussion because she realized that thinking about equivalent fractions in relation to converting fractions to decimals would allow her to extend her students' thinking should the opportunity arise. She recognized that emphasizing decimals as a "special kind of fraction with a power of 10 in the denominator" would help her students understand why some fractions terminated and why others did not. In retrospect, she wished that she had engaged the whole class in thinking about Miguel's approach so she could see how far they could go with it (see Kreith 2014 for a more extensive discussion of fig. 3).

FROM DIVISION PHOBIA TO UNDERSTANDING

Burnison's students had initially been division-phobic and were lacking number sense with respect to fractions and decimals. Some teachers might avoid exploring the relationship between fractions and decimals until after students had mastered long division and after students understood fractions better. Instead, Burnison had students explore terminating and repeating decimals; in so doing, it bolstered their abilities in division and their understanding of fractions simultaneously. Although Burnison addressed several practice standards, she continually adjusted instruction to take advantage of her students' contributions. She used the posters they produced as a resource and presented a student-generated question for all to investigate. She made it clear that she valued their work and found their ideas worthy of consideration. In response, students engaged in several SMP, actively looked for patterns, searched for structure, articulated their thinking, and attended to precision.

She was able to tailor instruction to her students' needs and input, meet the content standard, and facilitate the math practices because she had faith in her students' curiosity and in their intelligence. In addition, she explicitly valued their thinking. She had nurtured a classroom culture in which students felt comfortable sharing their thinking and questioning each other.

Students' conversations fueled their thinking, and together they came up with interesting ideas to pursue. This series of lessons felt like happy accidents that followed from students' ideas and questions. But really they were a result of lessons purposely designed to elicit student thinking which, in turn, provided further avenues for exploring math.

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Mediants Make (Number) Sense of Fraction Foibles

by Eric L. McDowell from Mathematics Teacher, August 2016

By the time they reach middle school, all students have been taught to add fractions. However, not all have learned to add fractions. The common mistake, of course, is to report that a/b + c/d is equal to (a + c)/(b + d). Teachers' knee-jerk reaction to this claim is to purge it from our students' minds. Perhaps we might illustrate that 1 + 1 would be equal to 1 if fractions were added this way: 1 + 1 = 1/1 + 1/1 = 2/2 = 1. Or we might indicate that the sum of 1/4 and 3/4 would be a value between these two fractions (i.e., 4/8 or 1/2), whereas the sum of two positive numbers must necessarily be greater than each.

It is certainly necessary to correct this mistake when a student makes it. However, this occasion also presents a valuable opportunity to enhance the student's mathematical confidence while also strengthening her number sense. Rather than placing all the emphasis on why her formula produces wrong answers, we should also acknowledge that she has "invented" an operation that is both useful and interesting. Regardless of her mathematical background, she will be proud and intrigued to learn that she has stumbled upon an operation that has utility in geometry, statistics, calculus, and other areas. In this article, we offer examples and activities that can be used to strengthen weaker students' basic numerical skills while honing the problem-solving abilities of the best and brightest. All students will be fascinated to see what can be reaped from examining this "mistake" in its proper contexts.

COURSE CONTEXT FOR THE MEDIANT

To distinguish this operation from ordinary addition of fractions, we will use the notation $\oplus$ to mean the operation defined by

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Formally, (a + c)/(b + d) is called the mediant of a/b and c/d. It has been studied by many mathematicians for many years. Indeed, references to the mediant are found in the writings of Plato as far back as 325 BCE; a thorough investigation of the history and study of the mediant can be found in Guthery (2011). Let's explore some applications of this operation in the settings of several mathematical areas.

Statistics

Suppose that a class consists of 12 boys and 18 girls. If 7 of the boys have brown hair and 8 of the girls have brown hair, then the ratios of brown-haired boys to all boys and brown-haired girls to all girls are 7/12 and 8/18, respectively. Because there are 15 brown-haired students in this class of 30, the ratio of brown-haired students to all students is 15/30. But 15/30 is exactly 7/12 $\oplus$ 8/18, which is the mediant of the two ratios.

Algebra

We generally use (a, b) to denote a vector with horizontal component a and vertical component b. The sum of vectors (a, b) and (c, d) is (a + c, b + d). So if we choose to represent vectors as fractions rather than as ordered pairs, then the sum of vectors a/b and c/d is a/b $\oplus$ c/d.

Coordinate Geometry

Consider the red line segment in figure 1 with positive slope a/b and the green segment with steeper positive slope c/d. As the diagram suggests, the orange line segment with slope a/b ⊕ c/d is necessarily steeper than the red segment and less steep than the green. As we will explore later, this will always be the case. Specifically, if a, b, c, and d are all positive and a/b < c/d, then it will always be true that a/b < a/b ⊕ c/d < c/d.

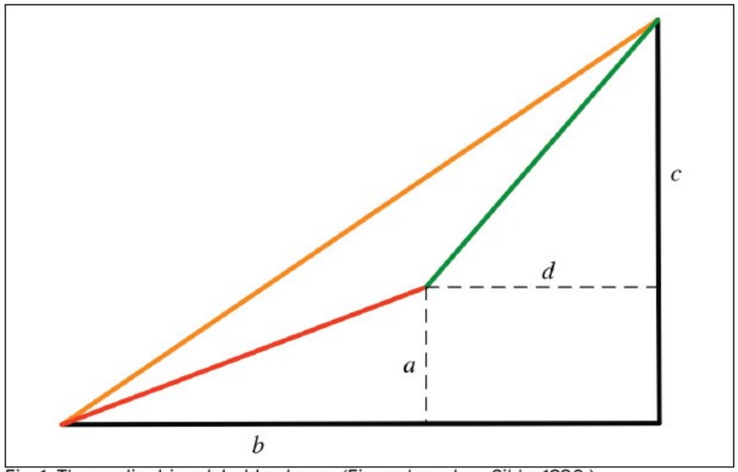

Fig. 1 The mediant is related to slopes. (Figure based on Gibbs 1990.)

Geometry

Suppose that we have two rectangles, one with area A and width w and another with area B and the same width w. (Assume that area and length are measured in square inches and inches, respectively.) Then the heights of these two rectangles are h1 = A/w and h2 = B/w. Observe that

$$\frac{A}{w} \oplus \frac{B}{w} = \frac{A+B}{w+w} = \frac{h_1 w + h_2 w}{2w} = \frac{\left(h_1 + h_2\right) w}{2w} = \frac{h_1 + h_2}{2},$$

which is the ordinary mean of the heights h1 and h2. This observation motivates us to define the average height of n rectangles with areas Ai and various widths w as

$$\frac{A_1}{w_1} \oplus \frac{A_2}{w_2} \oplus \cdots \oplus \frac{A_n}{w_n}.$$

Geometrically, this is the height of a rectangle with

$$\text{area } \sum_{i=1}^{n} A_i \text{ and width } \sum_{i=1}^{n} w_i.$$

As we will see, this definition for the average height of rectangles is a precursor to the usual notion of the average value of a function that we find in calculus.

Calculus

If f is a real-valued function defined on an interval [a, b], then the average value of f is defined as the (signed) height of the rectangle whose width is b – a and whose area is equal to the (signed) area between the graph of f and the x-axis. If

$$\sum_{i=1}^{n} f\left(x_i\right)\Delta x_i$$

is a Riemann sum of f on [a, b] that approximates this area, then the average value of f on [a, b] is approximated as the average (signed) height of n rectangles with (signed) areas f(xi) xi and widths xi for i = 1, …, n. As we saw in the

$$\frac{f\left(x_1\right)\Delta x_1}{\Delta x_1} \oplus \frac{f\left(x_2\right)\Delta x_2}{\Delta x_2} \oplus \cdots \oplus \frac{f\left(x_n\right)\Delta x_n}{\Delta x_n},$$

which can also be written as f x1 ( )Δx1 $$\frac{\sum_{i=1}^{n} f(x_i)\Delta x_i}{\sum_{i}^{n} \Delta x_i} = \frac{\sum_{i=1}^{n} f(x_i)\Delta x_i}{b-a}.$$

The limiting value of this expression is $$\frac{\int_a^b f(x)\,dx}{b-a},$$ which is the standard definition of the average value of f on [a, b].

ACTIVITIES INVOLVING THE MEDIANT

The mediant has a number of interesting properties, some of which are illustrated in the activities that are outlined in this section. Each activity is aligned with one or more of the Common Core State Standards for Mathematics (CCSSI 2010). The labels, numbers 1–8, will be used to identify the Common Core standards that are addressed by each activity.

1. Develop understanding of fractions as numbers. (Grade 3)
2. Extend understanding of fraction equivalence and ordering. (Grade 4)
3. Understand decimal notation for fractions and compare decimal fractions. (Grade 4)
4. Analyze and solve linear equations and pairs of simultaneous linear equations. (Grade 8)
5. Solve systems of equations. (High School: Algebra)
6. Perform arithmetic with polynomials and rational expressions. (High School: Algebra)
7. See structure in expressions. (High School: Algebra)
8. Reason with equations and inequalities. (High School: Algebra)

All the activities that are described in this section can be enjoyed by any student who has mastered standards 1–4 above. For some of these activities, extensions are suggested that address standards 5–8. Therefore, these activities should be accessible to all students in the upper grades, and the extensions should offer a sufficient degree of challenge to keep even the most talented students engaged.

ACTIVITY 1: FRIENDLY NEIGHBORS

Standards 1, 2, and 3 Ask students to write down all the fractions with values between 0 and 1 (inclusive) that have a denominator of 6 or less. Only fractions in lowest terms are to be included. Have students share their answers with one another until everyone is sure that his or her list is complete. Then have students use calculators to represent the value of each fraction in decimal form. Use these decimal representations to order the fractions from least to greatest. This ordered list (with decimal approximations included) is provided in table 1.

Table 1 An Ordered List of All Fractions between 0 and 1 with Denominator 6 or Less

0/1	1/6	1/5	1/4	1/3	2/5	1/2	3/5	2/3	3/4	4/5	5/6	1/1
0	0.16	0.20	0.25	0.33	0.40	0.50	0.60	0.66	0.75	0.80	0.83	1

Because both these factors are positive (recall that a/b is less than c/d by assumption), it follows that bc – ad > 0. With both the numerator and denominator positive, we conclude that the equivalent value (a/b $\oplus$ c/d) – (a/b) is positive. Showing that (c/d) – (a/b $\oplus$ c/d) is positive can be done similarly.

ACTIVITY 3: RECKONING RATIOS

Standard 2

Begin this activity by demonstrating how the mediant of two ratios can be used to represent a related third ratio, as we did earlier with the 8 brownhaired girls (out of 18 girls) and the 7 brown-haired boys (out of 12 boys). We found that the ratio of brown-haired students to total students in the classroom was 8/18 $\oplus$ 7/12, which is equal to 1/2. Point out that 8/18 is numerically equivalent to 4/9 but that the interpretation of the problem would be different if 4/9 were used in place of 8/18 (even though the percentage of brown-haired girls out of all girls would be unchanged). Ask students to consider what this difference would be. (Answer: There are now only 9 girls in the class, and 4 of them have brown hair.) Using the mediant, find the ratio of brown-haired students to total students in this new classroom. When they discover that 4/9 $\oplus$ 7/12 = 11/21, point out that this value is slightly more than 1/2. Experiment with other fractions that are numerically equivalent to either 8/18 or 7/12 (or both) to see how the resulting ratios change even when the percentages of brown-haired girls and brown-haired boys remain constant.

This activity illustrates that although $\oplus$ is an operation on the set of fractions (with positive denominators), it is not an operation on the set of rational numbers. Recall that an operation is a rule that assigns any pair of elements of a set to a unique member of that set. Because 4/9 and 8/18 are identical as rational numbers, any operation $*$ on the set of rational numbers would have to assign (4/9) $*$ (7/12) and (8/18) $*$ (7/12) to the same thing. However, 4/9 $\oplus$ 7/12 and 8/18 $\oplus$ 7/12 are not equal. This idea is illustrated in the next activity as well.

ACTIVITY 4: TARGET PRACTICE

Standards 2, 3, 4, and 5

Choose any four positive integers a, b, c, and d and pick any rational number that lies strictly between a/b and c/d. For example, 0.465 lies between 2/5 and 3/4. Now try to find fractions that are numerically equivalent to a/b and c/d whose mediant is equal to the chosen rational number. Continuing with our example, we see that 2/5 $\oplus$ 3/4 is equal to 5/9, or 0.555. . . . Too big. And 2/5 $\oplus$ 6/8 is about 0.615, which is bigger still. However, 4/10 $\oplus$ 3/4 is 0.5, which is heading in the right direction. And 8/20 $\oplus$ 3/4 is just a bit larger than 0.458. Let your students continue to search the (literally) endless combinations for a while before comparing answers. Then ask them to decide whose answer is closest to the target. Identify a winner and celebrate everyone's good work.

Extension 1

Although some students might approach this activity by selecting seemingly random equivalent fractions each time, others might develop a strategy that seems to produce better and better guesses. For example, when a guess is too large, multiplying the numerator and denominator of the lesser fraction by a common integer yields a mediant that is smaller (4/10 $\oplus$ 3/4 is less than 2/5 $\oplus$ 3/4). Similarly, multiplying the numerator and denominator of the larger fraction by a common integer yields a greater mediant. Watch for this and other potential strategies as students are working. At the conclusion of the activity, ask them to share their strategies.

Once you have verified that the students' lists are correct, point out that 2/5 is the mediant of its nearest neighbors, 1/3 and 1/2. Ask whether there are any other fractions on the lists with this property. Students should quickly identify 1/6, 3/5, denominators of 7 or less. (They can begin with the list already created and insert 1/7, 2/7, and so on into the appropriate locations.) Challenge students to verify this "neighboring" property for fractions with denominators of 10 or less.

ACTIVITY 2: WHERE'S THE MEDIANT?

Standards 1, 2, 6, 7, and 8 Choose any four positive integers a, b, c, and d and ask your students to calculate a/b ⊕ c/d. Then have them locate a/b, c/d, and a/b ⊕ c/d on a number line and write a number sentence that describes their numerical order using the less-than (<) sign properly. Have them repeat this exercise multiple times with different collections of four integers. Students should eventually recognize that the mediant always lies between a/b and c/d. (This fact was illustrated earlier in the example from coordinate geometry.)

Extension 1

This is actually an extension of activities 1 and 2 combined. Explain that the "betweenness" property of mediants can be used to build lists of rational numbers—in order from least to greatest—that can be represented as fractions with denominator n or less. Have students construct a line segment representing the interval [0, 1] that extends along the full length of a piece of paper and ask them to write down 0/1 above 0 at the far left and 1/1 above 1 at the far right. Calculate the mediant (1/2) and write it in its appropriate location between 0/1 and 1/1. Continue to include mediants of adjacent fractions (always reducing to lowest terms) until all adjacent fractions have mediants (in lowest terms) with denominator greater than n. When students think that they have finished, ask whether they can think of any fraction (in lowest terms) with denominator n or less that does not appear on their list. (If any exist, then there are still adjacent fractions whose mediant can be represented as a fraction with denominator n or less.) Once the lists are complete, have students observe that each fraction that they have written (other than 0/1 and 1/1) is the mediant of its neighbors.

Extension 2

For students with a stronger mathematical background, suggest that they try to prove that if a/b < c/d for positive integers a, b, c, and d, then it is necessarily true that

$$\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}.$$

One approach is to show that (a/b ⊕ c/d) – (a/b) and (c/d) – (a/b ⊕ c/d) are both positive. Note that

$$\begin{aligned}\left(\frac{a}{b} \oplus \frac{c}{d}\right) - \left(\frac{a}{b}\right) &= \left(\frac{a+c}{b+d}\right) - \left(\frac{a}{b}\right) \\ &= \frac{b\left(a+c\right) - a\left(b+d\right)}{b\left(b+d\right)} \\ &= \frac{bc-ad}{b\left(b+d\right)}.\end{aligned}$$

We already know that b(b + d) > 0 because the integers were chosen to be positive. To demonstrate that the numerator, bc – ad, is positive as well, write

$$bc-ad = \frac{bc-ad}{bd} \bullet bd = \left(\frac{c}{d} - \frac{a}{b}\right) \bullet bd.$$

Extension 2

It turns out that for any positive rational number p/q that lies strictly between a/b and c/d, there are fractions that are equivalent to a/b and c/d whose mediant is equivalent to p/q. Some stronger students might be able to prove specific cases of this fact if they are adept at solving simultaneous equations. We illustrate one possible proof for the example given above. To show that there are fractions equivalent to 2/5 and 3/4 whose mediant is equivalent to 0.465, find integers x and y for which bc−ad= bc−ad bd

$$\frac{2x}{5x} \oplus \frac{3y}{4y} = \frac{465}{1000}.$$

This is equivalent to solving the system 2x + 3y = 465 and 5x + 4y = 1000 for x and y. Doing so yields

$$x = \frac{1140}{7}$$
$$y = \frac{325}{7}.$$

Therefore, we have that

$$2\left(\frac{1140}{7}\right) + 3\left(\frac{325}{7}\right) = 465$$
$$5\left(\frac{1140}{7}\right) + 4\left(\frac{325}{7}\right) = 1000.$$

However, this is not quite what we were looking for: We were seeking integers that satisfied the system of equations. By multiplying both sides of the equations above by 7, we arrive at

2(1140) + 3(325) = 3255

5(1140) + 4(325) = 7000.

So it follows that

$$\frac{2(1140)}{5(1140)} \oplus \frac{3(325)}{4(325)} = \frac{3255}{7000} = \frac{465}{1000} = 0.465,$$

as desired.

EXPLORE THE "ERROR"

Playing with mediants is fun. Their properties are surprising, interesting, and engaging. Moreover, they offer ample opportunity to enhance the number sense of students at every level of ability. So when your students inevitably confuse a/b + c/d with a/b $\oplus$ c/d, introduce them to one or more of these activities. On second thought, why wait? By exploring properties of the mediant, students will have a concrete understanding of what this "error" really produces, and this deeper understanding is likely to encourage them to add fractions properly in the future.

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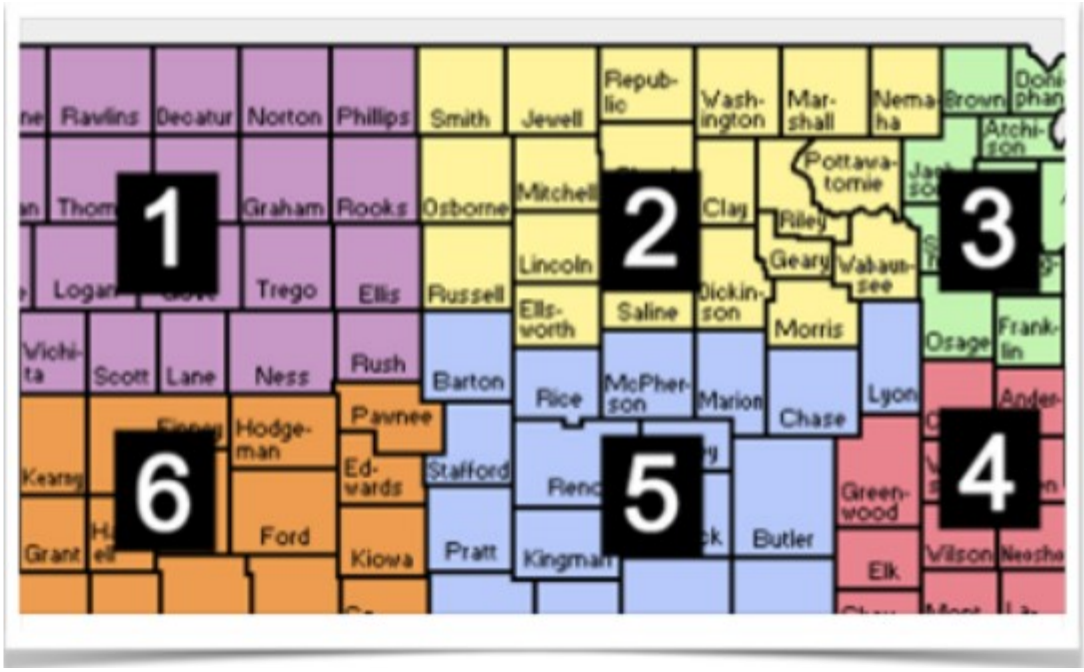
NOVEMBER 7, 2016

KATM: Zone 3 Update

Zone 3 Rep: Stacey Bell (staceybell@katm.org)



Stacey Bell

I am excited to serve as the Zone 3 Rep.

Meet Zone 3 Rep, Stacey Bell

Stacey Bell has been on the KATM Board for 4 years.

Stacey works as instructional coach at Shawnee Heights Middle School in Tecumseh, KS. She most recently served as Past-President. Before becoming an instructional coach, she was a 7th grade math teacher. She also serves on two NCTM Committees. Below are her goals for this new position. Feel free to email me with ideas or questions or needs that we can address!

1

PROFESSIONAL DEVELOPMENT

What are the needs of Zone 3 teachers?

2

NETWORKING

It is important to network with as many Kansas Teachers as possible.

3

GROW KATM MEMBERSHIP

Help teachers find value in KATM!

KATM Facebook

Join us on Facebook: Kansas Association of Teachers of Mathematics

KATM Twitter

Follow us on Twitter: KATM @KATMWebmaster

Goals for Kansas Teachers

Professional Development, Networking, and Growing KATM.

Being involved in Professional Organizations is important for many reasons. KATM allows teachers from all over the state of Kansas to network, learn from each other and have their voices heard in the State Board of Education as well as the State Legislature.

It has never been more important to come together to share ideas with each other and discuss the value of resources available to teachers. With the vast amount of information available on the internet, teachers need a way to know what resources are valuable and which are not. Getting involved in KATM is one way to do just that!

We are always looking for people to serve on our KATM Board! In addition, we are also looking for ways to get more teachers to see the value in joining KATM. This year, we had special prizes for KATM members at our Annual

Conference and placed our KATM Bulletin in a member's only location on our website. The Bulletin also includes special reprint articles from NCTM (permission granted) that you can read without paying for a full NCTM membership. We will have a call for nominations soon!

"We are always looking for people to serve on our KATM Board!"

In future newsletters, we will start to highlight the great teaching going on in our Zone, discuss upcoming professional development and take a look at resources available to teachers!

KATM: PROVIDING PROFESSIONAL DEVELOPMENT FOR KANSAS

Each year, we host an Annual Math Conference. For the last two years, we have held this conference in Maize, KS. Great News! Our next conference, will be held here in **Zone 3 at Seaman Middle School in Topeka, KS on October, 16, 2017.** We are just now starting the planning process. Stay tuned for more information on this conference!

What are you must haves in a conference? We would like to know. Previous keynote speakers have included: Jenny Bay-Williams, Kim Sutton, Greg Tang, and Chris Shore.

Call for Nominations!!!

The KATM Board is currently taking nominations to fill the following positions in the upcoming Board election. We are looking for educators that are interested in taking a leadership role in the field of Math Education throughout the State of Kansas. You can nominate yourself or someone that you know that has demonstrated a passion for advancing math in our state as well as someone that has a lot to offer in the way of supporting teachers. Please email Fred Hollingshead, Past President (hollingsheadf@usd450.net), with nominations and contact info of the nominee or fill out the online nomination form found at katm.org. Regular members in good standing are eligible for positions on the KATM Board. Nominations need to be completed by February. Elections will be held online at www.katm.org in March. A notice will be sent to remind you to vote.

Positions available for the upcoming election:

President-elect * 4-year term

The president-elect will serve for one year before then becoming president for a year, and then past-president for two years. The president-elect will assume the duties of president when needed. As president, the elected individual will preside over all KATM events and business meetings. The president will conduct the business of KATM as directed by the Executive Board and will represent KATM at a variety of functions, meetings, and conferences. The president is responsible for the overall functioning of the organization with assistance from the officers and Board members. As the past-president taking office in even-numbered years, this position will serve as the community relations representative for 2 years. This person shall be responsible for assuring communication between the Association and legislative, executive, and administrative branches of the government of Kansas.

Vice President – College * 2-year term

The vice president for college will attend all meetings and conferences and will assist the president in conducting the business of KATM. Each vice president will also encourage membership and will promote issues of special interest to their level represented in addition to serving on various committees as assigned. The person elected to this position will act as a liaison to college teachers.

Vice President – Middle School * 2-year term

The vice president for middle school will attend all meetings and conferences and will assist the president in conducting the business of KATM. Each vice president will also encourage membership and will promote issues of special interest to their level represented in addition to serving on various committees as assigned. The person elected to this position will act as a liaison to middle school teachers.

KATM Cecile Beougher Scholarship
ONLY FOR ELEMENTARY TEACHERS!!

A scholarship in memory of Cecile Beougher will to be awarded to a practicing Kansas elementary (K-6) teacher for professional development in mathematics, mathematics education, and/or mathematics materials needed in the classroom. This could include attendance at a local, regional, national, state, or online conference/workshop; enrollment fees for course work, and/or math related classroom materials/supplies.

The value of the scholarship upon selection is up to $1000:

- To defray the costs of registration fees, substitute costs, tuition, books etc.,
- For reimbursement of purchase of mathematics materials/supplies for the classroom

An itemized request for funds is required. (for clarity)

REQUIREMENTS:

The successful candidate will meet the following criteria:

- Have a continuing contract for the next school year as a practicing Kansas elementary (K-6) teacher.
- Current member of KATM (if you are not a member, you may join by going to www.katm.org. The cost of a one-year membership is $15)

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than June 1 of the current year:

1. A letter from the applicant addressing the following: a reflection on how the conference, workshop, or course will help your teaching, being specific about the when and what of the session, and how you plan to promote mathematics in the future.
2. Two letters of recommendation/support (one from an administrator and one from a colleague).
3. A budget outline of how the scholarship money will be spent.

Notification of status of the scholarship will be made by July 15 of the current year. Please plan to attend the KATM annual conference to receive your scholarship. Also, please plan to participate in the conference.

SUBMIT MATERIALS TO:

Betsy Wiens
2201 SE 53rd Street
Topeka, Kansas 66609

Go to www.katm.org for more guidance on this scholarship

Capitol Federal Mathematics Teaching Enhancement Scholarship

Capitol Federal Savings and the Kansas Association of Teachers of Mathematics (KATM) have established a scholarship to be awarded to a practicing Kansas (K-12) teacher for the best mathematics teaching enhancement proposal. The scholarship is $1000 to be awarded at the annual KATM conference. The scholarship is competitive with the winning proposal determined by the Executive Council of KATM.

PROPOSAL GUIDELINES:

The winning proposal will be the best plan submitted involving the enhancement of mathematics teaching. Proposals may include, but are not limited to, continuing mathematics education, conference or workshop attendance, or any other improvement of mathematics teaching opportunity. The 1-2 page typed proposal should include

- A complete description of the mathematics teaching opportunity you plan to embark upon.
- An outline of how the funds will be used.

An explanation of how this opportunity will enhance your teaching of mathematics.

REQUIREMENTS:

The successful applicant will meet the following criteria:

- Have a continuing contract for the next school year in a Kansas school.
- Teach mathematics during the current year.

Be present to accept the award at the annual KATM Conference.

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than **June 1 of the current year.**

- A 1-2 page proposal as described above.

Two letters of recommendation, one from an administrator and one from a teaching colleague.

PLEASE SUBMIT MATERIALS TO:

Betsy Wiens, Phone: (785) 862-9433, 2201 SE 53rd Street, Topeka, Kansas, 66609

KATM Board Meeting October Minutes

The KATM Board met for the fall 2016 meeting on Thursday, October 13 at Maize High School. The Board reviewed the agenda for the KATM Fall Conference and discussed the helping with registration.

Rhonda Willis from Pittsburg High School was appointed as the Zone 2 Representative, and Stacey Bell was appointed as the Zone 3 Representative.

Betsy Wiens reported that KATM membership has increased. Twelve KATM memberships were given away as door prizes at the fall conference.

In 2017, the Board will elect a President-Elect, a Vice President for Middle School and a Vice President for College.

The 2016-2017 KATM Bulletins will focus on the Progressions. In addition, there will be hyperlinks in the Bulletin to make it more user-friendly.

We received an update on the fall 2017 conference to be held at Seaman High School in North Topeka. The Board is soliciting sites for the fall 2018 conference.

Melissa Fast, KSDE Representative, gave the Board an update on changes to the Kansas College and Career Standards.

KATM Conference Report

Our fall conference was October 14, 2016 in Maize, Kansas. Our keynote speaker was Chris Shore. We had over 350 teachers register to attend, and almost 50 pre-service teachers. With over 30 presenters, it was a busy day!

The schedule for the day was split. Elementary began the day with a half day session with Chris Shore. Meanwhile, secondary teachers attended breakout out sessions. There was a wide variety of sessions including: student-centered math, using manipulatives, breakouts in the classroom, acceleration, several technology sessions, mathematical modeling and productive struggle.

In the afternoon, the schedule shifted. Elementary attended breakout sessions while secondary spent time with our keynote speaker. Elementary breakout session topics ranged from breakouts in the classroom, guided math, measurement with Mondrian, practice standards, computation, real-world math, problem-solving and student talk.

KATM Award Winners

Ray Kurtz Award

1995 Ray Kurtz
1997 Ruth Harbin Miles
1998 Kim Gattis
1999 William D Hammers
2000 Richard Driver
2002 Ethel Edwards
2004 Coltharp Clan
Forrest, Glenn, Hazel
2006 George Abel, Margie Hill,
Betsy Wiens
2007 Susan Gay, Sue Neal
2011 Melisa Hancock
2012 Allen Sylvester
2014 Germaine Taggart

*Cecile Beougher Award

1995 no applicants
1996 Tonya Polart
Kendra Schoeman
1997 Cheryl Rader
1998 reword
1999-2005 no applicants
2006 Lori Thompson
2007 no applicants
2008 Deb Nauerth
2009 David Fernkopf
2010 Jamie Junker
2011 Marianne Steen
2012 Amy Johnston
2013 no awardee
2014 Amy Johnston
2015 Jenny Howard
2016 Paula Ackerman

*Capitol Federal Award

2001 Judy Brummer
2002 Lorey Drieling
2003 Marlene Taylor
2004 Donna Young
2005 Rosabel Flax
2006 Adrianne Miller
2007 Patti Herbster
2008 Kathy Clouston
2009 Caprice Schaffer
2010 Jenny Wilcox
2011 Stephanie Vopat
2012 Washburn Rural Middle
2013 Julie Conrad
2014 Jamie George Provost
2015 JaLynn Urban
2016-David Leib

NCTM Update

My name is Stacey Bell and I am pleased to be the NCTM Rep for KATM. NCTM has a new website design and has been focusing on developing its Affiliate Site for its members. As an affiliate of NCTM, KATM is able to now post our upcoming events on this new site for neighboring states to see. And likewise, we are able to see what other affiliates are doing around us. You should check it out at http://www.nctm.org/affiliates/

In other news, the NCTM election results are in.

President-Elect

Robert Q. Berry III, University of Virginia, Charlottesville, VA

Director, High School Level

David Ebert, Oregon High School, Oregon, WI

Director, At-Large

Linda Ruiz Davenport, Boston Public Schools, Boston, MA

DeAnn Huinker, University of Wisconsin Milwaukee, Milwaukee, WI

Daniel J. Teague, North Carolina School of Science and Mathematics, Durham, NC

Being members of both NCTM and KATM allows you to have a wealth of resources for you and your classroom. For this issue, I would like to highlight some video resources found on NCTM's website about why conceptual teaching is so important and other videos related to the Common Core. Below is a description from NCTM's website. (http://www.nctm.org/Standards-and-Positions/Common-Core-State-Standards/Teaching-and-Learning-Mathematics-with-the-Common-Core/)

NCTM and The Hunt Institute have produced a series of videos to enhance understanding of the mathematics that students need to succeed in college, life, and careers. Beginning in the primary grades, the videos address the importance of developing a solid foundation for algebra, as well as laying the groundwork for calculus and other post-secondary mathematics coursework. The series also covers the Standards for Mathematical Practice elaborated in the Common Core State Standards for Mathematics and examines why developing conceptual understanding requires a different approach to teaching and learning.

- Building Conceptual Understanding for Mathematics
- Mathematics in the Early Grades
- Developing Mathematical Skills in Upper Elementary Grades
- Mathematical Foundations for Success in Algebra
- Preparation for Higher Level Mathematics
- Standards for Mathematical Practice
- Parents Supporting Mathematical Learning
- Conversations about K-12 Mathematics Education (Five-Part Series)

These videos will be helpful for all stakeholders, including parents. I encourage you to check them out.

Get Connected with KATM and Fellow Educators On Facebook & Twitter

Kansas Association of Teachers of Mathematics

twitter **@KATMWebmaster**

- Keep updated on Conference Updates.
- Keep updated on information about TWO Scholarships we give out each year!
- Keep updated about the changes to Kansas Math Standards.
- Talk with other educators about issues facing math educators.

Join our Facebook Group and follow us on twitter!

Do you like what you find in this Bulletin? Would you like to receive more Bulletins, as well as other benefits?

Consider becoming a member of KATM.

For just $15 a year, you can become a member of KATM and have the Bulletin e-mailed to you as soon as it becomes available. KATM publishes 4 Bulletins a year. In addition, as a KATM member, you can apply for two different $1000 scholarship.

Current members—-refer three new members and you get one free year of membership!

Join us today!!! Complete the form below and send it with your check payable to

KATM to:
Margie Hill
KATM-Membership
15735 Antioch Road
Overland Park, Kansas 66221

Name________________________________

Address______________________________

City_________________________________

State________________________________

Zip__________________________________

Home Phone__________________________

HOME or PERSONAL EMAIL:

____

Are you a member of NCTM? Yes___ No___
Position: (Cirlce only one)
Parent
Teacher:: Level(s)________
Dept. Chair
Supervisor
Other

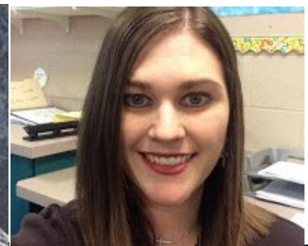
Referred by: __________________________________

____

KANSAS ASSOCIATION MEMBERSHIPS
Individual Membership: $15/yr. ___
Three Years: $40 ___
Student Membership: $ 5/yr. ___
Institutional Membership: $25/yr. ___
Retired Teacher Membership: $ 5/yr. ___
First Year Teacher Membership:$5/yr. ___
Spousal Membership: $ 5/yr. ___
(open to spouses of current members who hold a regular Individual Membership in KATM)

KATM Executive Board Members

President: David Fernkopf, Principal, Overbrook Attendance Center, dferkopf at usd434.us



President: Elect: Stacey Ryan, Middle School teacher, Anodover Middle School, ryans@usd385.org

Past President, NCTM Rep: Stacey Bell, Instructional Coach, Shawnee Heights Middle School

bells at usd450.net, 785-379-5830

Past President, Community Relations: Pat Foster

Principal, Oskaloosa Elementary School

pfoster at usd341.org

Secretary: Janet Stramel, Assistant Professor, Fort Hays State Univ.

jkstramel at fhsu.edu

Vice President, College: Lanee Young

Membership Co-chairs: Margie Hill, Instructor, Kansas University

marghill at @ ku.edu

Vice President High School: Amber Hauptman, Math Teacher, Washburn Rural High School

hauptamb at usd437.com

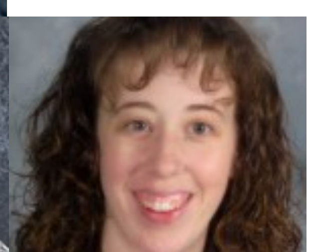
Membership Co-Chair: Betsy Wiens, Math Consultant

betsy.wiens at gmail.com

Vice President Middle School: Blake Carlson, Middle School Math Teacher, French Middle School

bcarlson at tps501.org

Treasurer: David Fernkopf, Principal, Overbrook Attendance Center, dferkopf at usd434.us



Vice President Elementary: Amy Johnston, 2nd grade Teacher, Auburn Elementary

johnsamy at usd437.com

KSDE Liaison: Melissa Fast, Math Education Consultant

mfast at ksde.org

Bulletin Editor: Jenny Wilcox, 7th grade teacher, Washburn Rural Middle School,

wilcojen at usd437.net

KATM Executive Board Members

Zone 1 Coordinator:

Jerry Braun, Hays Middle School, jj_ks at yahoo.com

Zone 4 Coordinator:

This position is currently open.

Zone 2 Coordinator:

Kira Pearce

Zone 5 Coordinator:

Lisa Lajoie-Smith, Instructional Consultant, llajoie at sped618.org

Zone 3 Coordinator:

Stacey Bell

Zone 6 Coordinator:

Jeanett Moore, 2nd grade teacher, USD 48

Jeanett.moore at usd480.net

Webmaster: Fred Hollingshead
Instructional Coach, Shawnee Heights High School
hollingsheadf at usd450.net