KATM Bulletin

Kansas Association of Teachers of Mathematics

February 2016

Reason abstractly and quantitatively

Inside this issue:

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Save the Date: October 14, 2016

Maize High School

KATM Annual Conference

Hope to see you there!

A Message from our President

Hello Kansas mathematicians! In this edition of the bulletin, we highlight mathematical practice #2 which asks students to reason abstractly and quantitatively. To be proficient in this practice, students should be able to reason using models of pictorial representations, translate events into symbols for solving problems, and appropriately solve problems using the symbols. One way to help students use this mathematical practice is to have them think about the problem in their head before they begin solving the problem using other materials. Helping them develop the attitude that they can solve the problem in more than one way will also be a benefit for students. Finally, have students use numbers and words interchangeably to help make sense of a problem is also a good strategy for this mathematical practice.

Along with highlighting this important practice, I would also like to take this opportunity to encourage Kansas math teachers to become more actively involved in their professional organization. We are always looking for people to run for elected office or volunteer to be nominated for a position on the board. Currently, we need representatives from two zones across the state. Zone 1 represents Northwest Kansas and Zone 4 represents Southeast Kansas. Both of these zones lack representation on the KATM Board. Zone coordinators help plan professional learning events in their region and promote membership interest in their zone. If you are interested in being a Zone Coordinator or know of some who would be interested, don't hesitate to contact a current member of the board. Thanks for your commitment to Kansas kids and math education in our great state!

Pat Foster

Patrick Foster

President, KATM
president@katm.org

Hello Kansas Math Teachers! I hope this Bulletin finds you well, and not too stressed out as the spring approaches! I can't believe we're almost done with our Math Practices series. I'm looking forward to a new series of Bulletins based on content standards....seems like a natural follow-up to the math practices focus. I would love to hear from you....what would you like to see in our Bulletin? What are the most useful things we can include? Email me your thoughts! jennywilcox@katm.org

Jenny Wilcox, KATM Bulletin Editor

In Coming Issues

April 2016—This issue will wrap up our series of issues on the standards for mathematical practice. We will focus on practices 7 and 8 in this issue, look for and make use of structure and look for and express regularity in repeated reasoning.

Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

As we conclude our Bulletin series on mathematical practices, we look forward to starting a new series of Bulletins. Thoughts about what you'd like to see us focus on? We'd love to hear from you!

#2 Reason abstractly and quantitatively

Summary of Standards for Mathematical Practice

- Make sense of quantities and their relationships.
- Are able to decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the means of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attends to the meaning of quantities, not just how to compute them.

Questions to Develop Mathematical Thinking

- What do the numbers used in the problem represent?
- What is the relationship of the quantities?
- How is _____ related to ______?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might we use to find a solution?
- How did you decide in this task that you needed to use…..? Could we have used another operation or property to solve this task? Why or why not?

Teacher Actions/Responsibilities

- Create problems such as "What word problem will this equation solve?"
- Give real-world situations.
- Place less emphasis on the answer.
- Provide a range of representations of math problem situations and encourages various solutions.
- Provides opportunities for students to make sense of quantities and their relationships in problem situations.
- Provides problems that require flexible use of properties of operations and objects.
- Emphasizes quantitative reasoning habits of creating coherent representations of the problem at hand; considering the units involves; attending to the meaning of quantities, not just how to compute them and/or rules; and knowing and flexibly using different properties of operations and objects

Student Actions/Responsibilities

- Represent abstract and contextual situations symbolically.
- Use varied representations and approaches when solving problem.
- Make connections, including real-life situations.
- Visualize problems.
- Make sense of quantities and their relationships in problem situations.
- Are decontextualizing (abstract a given situation and represent symbolically and manipulate the representing symbols), and contextualizing (pause as needed during the manipulation process in order to probe into the referents for the symbols involved.
- Use quantitative reasoning that entails creating a coherent representation of the problem at hand, considering the units involved, and attending to the meaning of quantities, NOT just how to compute them

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences

Teacher: actions that further the development of math practices within their students

Task:

- Includes questions that require students to attend to the meaning of quantities and their relationship, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

Teacher:

- Asks students to explain the meaning of the symbols in the problem and in their solution.
- Expects students to give meaning to all quantities in the task.
- Questions students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

We would love to hear from you! How do you get your students to reason abstractly and quantitatively? What questions do you ask that help students with this mathematical practice? What lesson do you have that highlights this mathematical practice?

Our new website now allows for online Bulletin submissions. Try it out!

Toy Stories: Modeling Rates

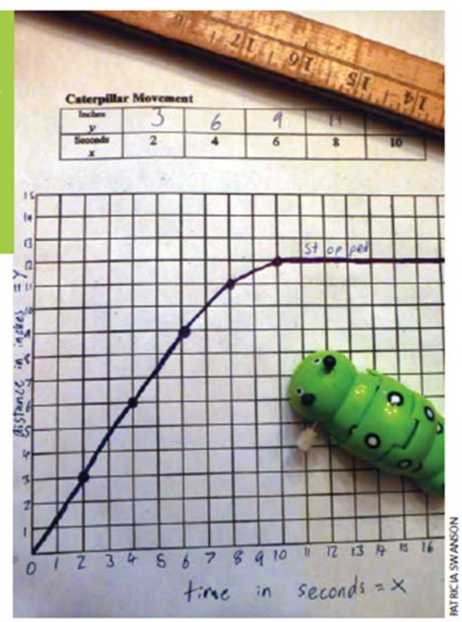
by Patricia Swanson, from September 2015, Teaching Children Mathematics

Elementary school mathematics is increasingly recognized for its crucial role in developing the foundational skills and understandings for algebra. Among the most challenging of these concepts is the relationship between two variables—covariation—as represented by a series of ratios comparing two quantities, paired values on a T-table, or points on a graph. Each of these representations tells a story of two quantities, or measures, and how they are related; but the story is poorly understood by many students (Lobato and Thanheiser 2002).

I explored this issue last year, researching the teaching of critical foundations of algebra (National Mathematics Advisory Panel 2008) in a K–grade 8 one-room rural school. Although I worked with third through eighth graders, this lesson was designed primarily for the fourth through sixth graders as a lead-in to understanding rates and how to show them on a graph.

Rates are a special kind of ratio—whereas ratios compare two quantities in a given situation, rates compare two different types of measures, in different units (Van de Walle, Karp, and Bay-Williams 2010). Comparing the number of miles to hours traveled is a rate because it compares different kinds of measures. Rates often describe how quantities change over time (Lamon 2005). When rates are expressed as a comparison to a single unit, they are called *unit rates*—for example, the number of miles traveled in just one hour, or the cost for a single item (Chapin and Johnson 2006). Students are taught to calculate unit rates by dividing the first term by the second; for example, if a car travels 100 miles in 2 hours, students divide 100 by 2 to calculate the unit rate of 50 miles per hour.

This lesson introduces the concept of rates—comparing two different types and units of measure—and how to graph them.



In this lesson, I chose the concept of speed, a rate comparing distance over time, to present students with a concrete, hands-on experience in which they could gather data on two measures, or variables, and explore their relationship. We quickly discovered that the lesson would provide a crucial reference point for an array of subsequent lessons focused on rates, proportional reasoning, graphing, and speed as represented by the slope of a line. But it was one student's question as she struggled to make sense of the numbers that demonstrated the value of the lesson and its potential to engage students in practices central to the discipline of mathematics.

In this article, I describe the lesson and share materials I developed to teach it. I then use the student's question as a lens to discuss the mathematical practices that students engaged in as they attempted to make sense of their data. This lesson illustrates how we can put into place key mathematical practices identified in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010).

The lesson

We collected data on the speed of small windup toys, measuring distance traveled over time. We discussed that as mathematicians, we could measure speed as a rate, in this case comparing distance measured in inches to time measured in seconds. By measuring the total distance traveled by a toy at a series of specified times, we could represent our data as a series of rates, or organized in a T-table showing *x*- and *y*-coordinates representing time and distance. These coordinates locate points on a line graph. The slope of the line segments connecting the points represents the rate we call *speed*. Each of these representations is a tool we use to model with mathematics, mapping the relationships between quantities to interpret and draw conclusions (CCSSI 2010). I asked students to compare these tools and decide which they believed best modeled their observations of the toys' movement.

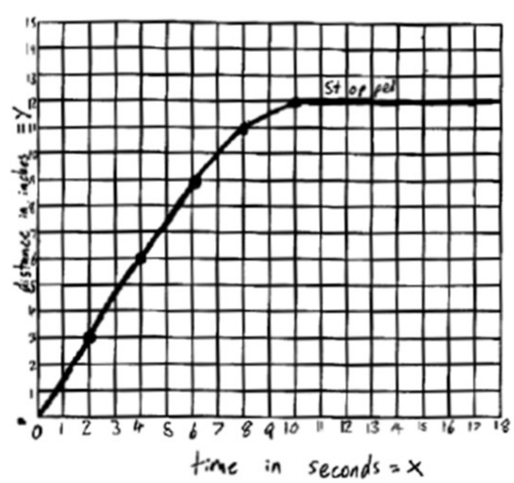
We first practiced with a delightfully slow and steady caterpillar toy, recording our data as a series of rates expressed as distance over time (see fig. 1). We organized consecutive time measures on a table showing the total distance the caterpillar traveled in two, four, six, eight, and ten seconds. Students were familiar with graphing points on a coordinate grid, so we used our tables to create ordered pairs that could be graphed. We represented time on the *x*-axis and distance on the *y*-axis. As guided practice, each student recorded the caterpillar data in his or her own table and graphed the points showing distance over time. We discussed the line graph, noting that the first three points on the graph reflected the caterpillar apparently traveling at a constant rate of three inches for every two-second interval. However, after six seconds, the toy slowed, and as its speed decreased, the slope of the line segments connecting the next two points changed and became less steep. Finally, after ten seconds, the caterpillar stopped, and the line became horizontal, showing that the toy gained no further distance as time passed.

FIGURE 1

Taken from a student activity sheet, this table shows a series of caterpillar rates expressed as distance over time. The student then graphed the table data as ordered pairs.

Caterpillar Movement

Inches y	3	6	9	11	12	stopped
Seconds x	2	4	6	8	10	



Students noted that our line graph did not look like most line graphs in their textbook, showing one line representing a constant rate. Rather, our line graph was in line segments, the slope of each representing the toy's speed between consecutive time measures. I referred to our graph as *a mathematical representation modeling the toy's progress over time*, but students called it a *toystory line*—a graph with a story to tell.

I assigned students to work in pairs or triads, giving each group a bag with two or three windup toys to compare. Each group had a task card outlining the group project and individual activity sheets on which to record their data and explain their findings (see online activity sheets 1 and 2).

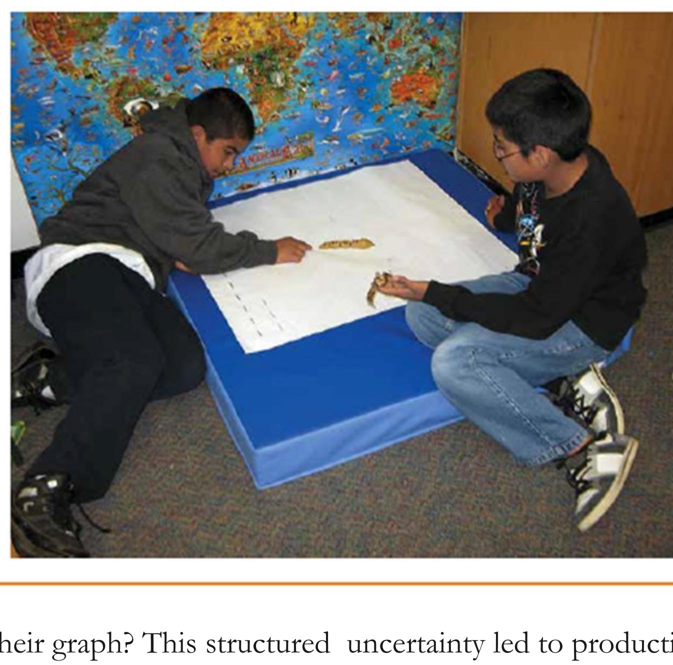
Group work

Although students found the task highly engaging, it carried its share of pitfalls. Students predictably fixated on the toys, not the math. They were determined to find out which toys were fastest by racing them rather than measuring distance over time. In addition, the toys' movement was not always easy to measure. Some traveled quickly; others stalled out or required periodic nudges to travel in a straight line. In retrospect, I should have selected the windup toys more carefully. Slow, relatively steady, sturdy toys are best for this activity.

Nonetheless, the challenges of collecting data on toy rates led to rich discussions on the meaning of rates and the importance of understanding them in context (Lamon 2005). The wind-up toys illustrated that rates can be constant or varying, a key point often ignored in the student text. Indeed, "real" data on moving objects, coordinate points showing distance over time, rarely fall on a perfect line. In addition, although most texts at this level focus on extending rates proportionally, that computation would be misleading in this context. Rather than a series of proportional rates modeled by one single line, our line graphs showed a series of line segments (see fig. 2). The slopes of the line segments reflect the speed of the toy between two consecutive points in time. Ultimately we would need to bridge to the text and practice calculations assuming constant rates, but this activity set the stage for my students to critically examine the context of any rate problem. In subsequent lessons, when grappling with rate problems, our discussions always began with the meaning of the rate and the context in which it was used.

FIGURE 2

Measuring time and distance as a series of line segments —rather than of proportional rates as a single line— taught students to look for the context of rate problems.



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In spite of our rather rocky beginning, groups ultimately settled into the task. I had purposely designed the task so that students would need to discuss and decide a number of essential issues. For example, at what time intervals would they measure distance? How would they scale their graph? This structured uncertainty led to productive group conversations (Lotan 2003). Because I structured the groups to be academically heterogeneous, these math discussions also involved considerable helping and teaching. For example, in one group, a fifth grader who was trying to capture the rapid movement of the group's speeding wind-up bird (see fig. 3) grappled with how to scale distance on the graph (see fig. 4) by dividing the greatest distance the bird traveled by the number of squares on her graph. She then taught her younger partner how to skip-count correctly to label the *y*-axis of her graph.

FIGURE 3 Fifth-grade students, trying to convey a wind-up bird's speed, scaled distance on a graph by dividing the greatest distance the bird traveled by the number of squares on the graph.

PATRICIA SWANSON

FIGURE 4 One of the fifth graders taught her younger partner how to skip-count correctly to label the *y*-axis of her graph.

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Wrap-up

When we gathered as a class, we discussed students' accurate data representations both on tables and line graphs. The graphs showed the toys' progress, or distance, at different points in time, and the lines connecting the points documented the toys' somewhat erratic speed.

As is often the case, the multifaceted mathematical potential of the lesson emerged during our wrap-up session, when we were discussing students' data. First, we used samples from students' data to practice calculating the unit rate or average speed of the toys at given points in time. By dividing the total distance traveled by the total time elapsed, we calculated the distance traveled in just one second. Note that the unit rate or average speed does not reflect (as many students think) the arithmetic mean of the different speeds. Rather, average speed implies *proportional distribution*, as if the toy had traveled at a constant rate for the total elapsed time (Lamon 2005). For example, on the caterpillar graph, the line segments' slopes reflect several different rates, or speeds, but the unit rate, or average speed, for the total elapsed time of ten seconds assumes a constant speed calculated at 1.2 inches per second.

We selected rates from students' data tables to practice. I started with "friendly" numbers. For example, the toy car traveled a total of 6 inches in three seconds; six divided by three gave us a unit rate, or average speed, of 2 inches per second. We practiced with increasingly difficult rates, the older students dividing distance by time to calculate the unit rates, and the younger students checking their answers with calculators. Using student data to calculate unit rates and practice computation gave students a stake in finding meaningful answers.

Sense making

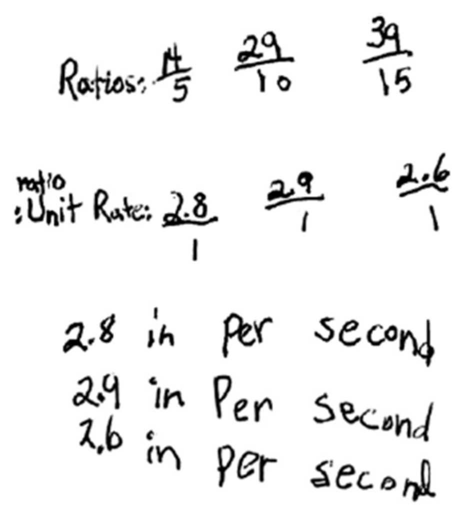
From these computations, an older student, Cecilia, who often asked *how* to do something in mathematics but rarely asked *why,* looked at her data (see fig. 5) and stated,

> This doesn't make any sense. The fractions are getting bigger, but the other, the unit rate—see here? It gets smaller.

Her question illustrates a serious misconception regarding fractions, as it is not the size of the numbers but the relationship of the part (the numerator) to the whole denominator) that indicates the size of the fraction. Similarly, the relationship between distance and time defines speed. Although understanding this relationship is critical, addressing Cecilia's struggle to make sense of her data was of first importance. Cecilia was engaging in the first of eight mathematical practices identified in the Common Core—to *make sense of problems and persevere in solving them* (CCSSI 2010). Cecilia rarely tried to make sense of mathematics. It was a milestone for her in that on this day, having collected the numbers herself, she thought the numbers ought to make sense.

FIGURE 5

When Cecilia's rate computations prompted her to question *why* the numbers made sense, she was engaging in the first of the Common Core's eight Standards for Mathematical Practice.



The National Research Council identified productive disposition as one of the five interrelated strands of mathematics proficiency. They defined productive disposition as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (Kilpatrick, Swafford, and Findell 2001, p. 116). Cecilia was developing a productive disposition—the expectation that mathematics ought to be sensible and that she was persisting.

I called the class together to explore Cecilia's question, first pointing out that her question showed she was engaging in the most important mathematical practice, to make sense of her data. Her attempt to link the toy's observed movement to the numbers she generated and her expectation that both her "fractions" and her unit rates should tell the same story, were commendable. She was thinking like a mathematician.

We used her question as an opportunity to engage in several other of the Common Core's Standards for Mathematical Practice (SMP) to make sense of the data: (1) to model with mathematics (SMP 4), (2) to look for and express regularity in repeated reasoning (SMP 8), and (3) to reason abstractly and *quantitatively* (SMP 2). We revisited our data on the caterpillar whose slow, steady movement yielded friendly numbers on the ratio table.

We once again observed the caterpillar's movement and crosschecked our first three rates on the table with points on the graph. I asked students if they could tell me how far the caterpillar went in just one second—not by dividing, but by looking at the number patterns on their tables or the line on the graph. In their groups, they once again discussed the caterpillar data, looking for regularity or patterns in the data that they could use to find the unit rate. Some used the rate table, reasoning that if the bug went 3 inches in two seconds, then it would go 1 1/2 inches in one second. Others noted that the line on the graph showed a distance of about 1 1/2 inches at the one-second mark on the *x*-axis. Finally, we examined Cecilia's question: Why did the numbers on the rate table get bigger when the unit rate at first stayed the same and then decreased? Students engaged in the second mathematical practice, to reason abstractly and quantitatively, as they divided to find unit rates and grappled with the question. They compared the increasing numbers on the ratio table with the initially constant but then declining unit rates for the caterpillar. They explained to one another why the numbers representing time and distance got bigger, but the unit rate, the relationship between the two numbers, did not. Their conversations led to one final example of modeling with mathematics.

Students noted that the steepness of the lines on their graphs reflected the toy's speed. In their words, "the steeper the line, the faster the toy." When the line flattened, the toy was slowing down. A horizontal line meant that the toy had stopped. The changing slopes of the line segments on their graphs modeled the story of the toy's speed over time. I challenged them to informally map—without numbers—the speeds of different toys, estimating the slopes of the lines to illustrate the toys' changing speed. With gusto, students gathered around a large table, observed each different toy travel across the floor, and sketched what its graph might look like. Although their informal line graphs were based on observation, not measurement, I prompted them to describe how each toy's changing speed could be modeled and interpreted with their informal graphs (see fig. 6).

Common Core Connections

SMP 1 SMP 2
SMP 4 6.RP.3

Traveling from concrete to abstract

Measuring rates with wind-up toys was not only fun but also a mathematically rich task. I often use highly engaging, concrete tasks such as this to serve as foundational activities for a unit of study. I believe these tasks merit the time, for they serve as a memorable grounding experience for students, a concrete example, and a reference point for subsequent lessons. If we are to help students travel from concrete to abstract representations, to move from an understanding of whole numbers to the challenges of rational numbers and proportional reasoning, then we must scaffold their understanding with such tasks as this one. It promoted substantive mathematical discussion and presented the opportunity for students to engage in the kinds of mathematical practices that foster deep engagement in the discipline of mathematics.

About the author

Patricia E. Swanson, **patricia.swanson@sjsu.edu**, teaches preservice and masters' level courses at San José State University, California. She recently developed the Advanced Teaching Certificate in Common Core Mathematics for practicing teachers and enjoys teaching and conducting research in a K–grade 8 rural schools.

"Toy Stories: Modeling Rates" activity sheet 1

Name ____________________

Task Cards

Your task is to mathematically represent the speed of your toys. In this lesson, you will use rates and a graph to compare the speeds of the different toys.

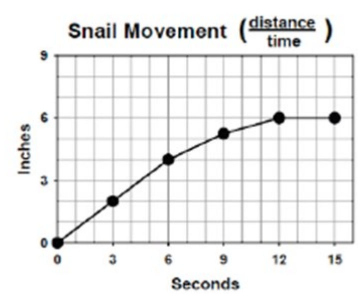
A rate is a ratio of two quantities measured in different units. Rates compare two measures. Speed is measured as a rate comparing distance and time. Rates can be written a number of ways.

15 miles per hour   15 mi/h $\frac{15 \text{ mi}}{1 \text{ hr}}$

If you know the total distance traveled and the total time it took to travel that distance, you can calculate the **unit rate**, or average speed, by dividing the distance by the time:

$$\text{Unit Rate (or Speed)} = \frac{\text{Distance}}{\text{Time}}$$

A **graph** can also be used to illustrate distance traveled at different points in time.



Design an experiment to test the speed of your toys. How far do they travel over time?

Represent your results as a series of rates at different points in time.

As a group, make a graph showing the distance each toy traveled over time. Compare the speed of the toys in your group. Prepare to present your group results with an explanation of how both the rates and your graph may be used to compare the speeds of the toys.

Caution: Do NOT overwind your toy or it will break!

→ "Toy Stories: Modeling Rates" activity sheet 2

Individual Report

Complete the rate table for your toy. You will need to decide on the time interval you wish to measure.

Example: Bumble Bee

Distance (inches)	6	11	17	20
Time (seconds)	3	6	9	12

Note: Your rates may not be proportional.

Toy:

Distance (inches)				
Time (seconds)				

Complete rate tables for the other toys in your group.

Toy:

Distance (inches)				
Time (seconds)				

Toy:

Distance (inches)				
Time (seconds)				

Which toy is the fastest? Justify your answer.

Why do you think the rates on your tables are not perfectly proportional?

A Balancing Act: Making Sense of Algebra

by Katherine Gavin and Linda Jensen Sheffield

We recently asked a group of middle school teachers, many of whom taught Algebra, to answer the question, "What is algebra?" After a pregnant pause, we heard, "solving equations", "determining a value for the unknown", "using variables", and "graphing equations". Although these answers are all part of doing algebra, this branch of mathematics encompasses much more.

For most students, algebra seems like a totally different subject than the number topics they studied in elementary school when in reality the procedures followed in arithmetic are actually based on the properties and laws of algebra. In fact, algebra should be a logical next step for students in extending the proficiencies they developed with number because "algebra is simply a language for exploring and explaining mathematical patterns" (Bressoud 2012, p. 1).

In *Adding It Up: Helping Children Learn Mathematics*, the National Research Council delineates two aspects of algebra as being "(a) a systematic way of expressing generality and abstraction including algebra as generalized arithmetic; and (b) a guided transformation of symbols such as we do when we solve equations by collecting like terms and using inverse operations" (NRC 2001, p. 256). These transformational aspects have traditionally been emphasized in such a way that algebra becomes a study of procedures and rules rather than an exploration of concepts that lead to generalizations that support the rules or make the equation or expression meaningful to the student. Research has shown that these rule-based approaches to teaching and learning lead to forgetting the rules (e.g., Kirshner and Awtry 2004), unsystematic errors (e.g., Booth 1984), reliance on visual cues (Kirshner 1989), and poor strategic decisions (e.g., Wenger 1987).

We need to help students develop and make sense of the rules they are using and show them how to employ a variety of strategies to solve algebraic problems. We also need to help students see algebra as generalizing computational procedures and operations they use with numbers. The Common Core State Standards for Mathematics (CCSSM) for middle school advocate for this in its discussion of expressions and equations for grade 6: "Apply and extend previous understandings of arithmetic to algebraic expressions" (CCSS 2010, p. 41). Activities for students should include opportunities to make sense of problems; reason abstractly; model their thinking using graphs, tables, diagrams, and so on; look for and make use of structure; and create arguments to justify their thinking and critique the reasoning of others. These are, in fact, four of the Common Core's Standards for Mathematical Practice (SMP).

We had an opportunity to work with students across the middle grades implementing a curriculum that encouraged students to think and act like mathematicians, thus using the Mathematical Practices consistently. We found that a deliberate emphasis on the Mathematical Practices while learning CCSSM content helped our students gain a much deeper understanding of concepts and procedures and the ability to generalize these using algebraic reasoning and notation.

EXPLORING EQUALITY AND BALANCE SCALES

We began in grade 6 with the concepts of equality and balance that are central to the study of equations. Research has shown that students need help constructing meaning for equality (e.g., Falkner, Levi, and Carpenter 1999; Kieran 1981; Saenz-Ludlow and Walgamuth 1998). Table 1 shows how students in grades 1 through 6 responded when asked what number they would put in the box to make the sentence $8 + 4 = \square + 5$ true.

Table 1 Students in various grades had different responses to make the sentence $8 + 4 = \square + 5$ true.

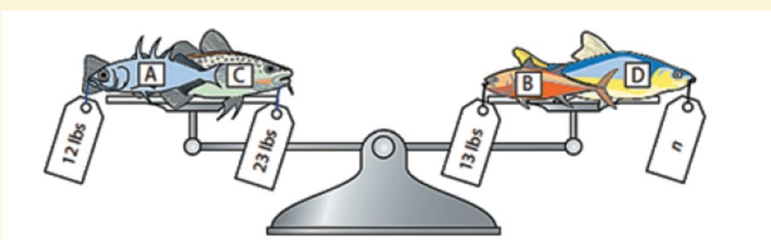
Responses	Grades		
	1 and 2	3 and 4	5 and 6
7	5%	9%	2%
12	58%	49%	76%
17	13%	25%	21%
12 and 17	8%	10%	2%

Source: Adapted from Falkner, Levi, and Carpenter 1999, p. 132

Note that a lower percentage of students in grades 5 and 6 (only 2 percent) got the correct answer compared with those in grades 1 and 2 (5 percent).

In the following investigation, students harkened back to ancient Egypt and imagined working in a fish market. They were shown the balance scale in **figure 1** and asked to write an equation for the weights of the fish on the scale. We asked them to use *n* in their equation to represent the unknown weight of the fish in pounds.

Fig. 1 Students were asked to write an equation and find the weight of the unknown fish in this diagram.



Source: Sheffield, Chapin, and Gavin 2010, p. 14

We also asked them to solve the equation for the missing weight without finding the total weight of the fish on the left. We introduced students to writing equations at this point but not to the typical rules in solving them. After giving students a chance to write their equations and think about how to find the weight of fish D, teachers and students engaged in the following conversation:

Ms. Jackson: KayAn, what did you write for your equation?

KayAn: I wrote $12 + 23 = 13 + n$. I could see that the fish on the left balanced the fish on the right. I used equals to show they were the same.

Ms. Jackson: Jason, can you repeat what KayAn said?

Jason: I think she said that she wrote 12 + 23 = 13 + *n* 'cause that is what I wrote. You can see the balance on the scale.

[Other students nodding.]

Ms. Jackson: It looks like we agree on the equation. How did you solve it, Sarella?

Sarella: I saw 13 is 1 more than 12, so I knew that *n* would be 1 less than 23.

Carey: If 13 is 1 more than 12, why isn't *n* 1 more than 23?

Sarella: Well, see . . . you have to make up for it by taking 1 off the 23.

Ms. Jackson: Everyone turn to your elbow partner and talk about Sarella's solution and Carey's question. Should you add 1 to 23 or subtract 1 from 23?

After a one-minute discussion, Ms. Jackson asked Jared to share his answer. He drew blocks on a balance scale on the white board. He took 1 block from the pile of 23 blocks on the left side of the scale and moved it to the pile of 12 blocks to make 13. He then had one pile of 13 blocks and one pile of 22 blocks on the left side of the balance scale. He said that the pile of 13 blocks on the right side had to match the pile of 13 blocks on the left, leaving the unknown pile to be 22 blocks. Agreement was reached that this made sense.

Ms. Jackson: Did anyone solve it another way?

DeShawn: I saw that 23 blocks is 10 more than 13. That meant *n* is 22.

Susana: Where did 22 come from?

DeShawn: [Pointing to the fish] See this fish B with 13 pounds; it is 10 pounds less than this fish C. That means that fish D has to make up by adding 10 pounds. See, it has to be 10 pounds more than A or it won't balance.

Students figured out the solution without using pencil or paper. They made sense of the problem, explained and justified their methods, critiqued one another's reasoning, and gained a stronger grasp of the meaning of equality and how to interpret the equals sign (SMP 3).

PROPOSING BAR DIAGRAMS AS EFFECTIVE TOOLS

After solving a few more fish-weighing problems, students were exposed to a different solution method. In the example that we posed, we stated that Mei Ling had drawn a bar diagram, shown in **figure 2**, to solve the problem. We asked students to explain how Mei Ling might have used this diagram to find the unknown weight.

Fig. 2 This bar diagram illustrates another model to represent the fish problem.

12 pounds	23 pounds
13 pounds	*n* pounds

Source: Sheffield, Chapin, and Gavin 2010, p. 15

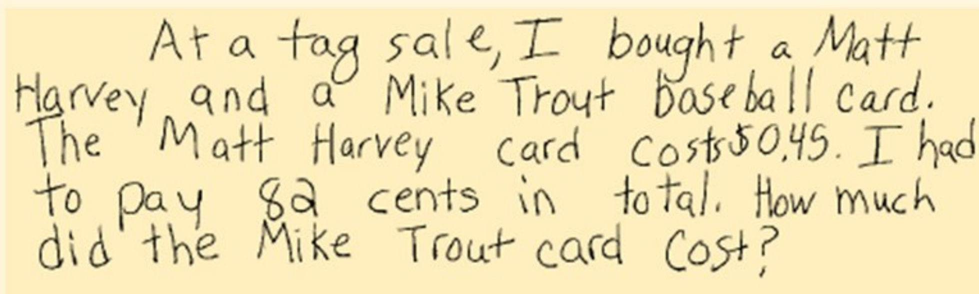
Some students found this visual easier to understand than the balance scales. They saw that the total of 13 pounds was a little more than 12 pounds, so they needed to subtract from 23 pounds to get the value of *n*. English language learners particularly benefited from this visual model. Throughout the investigation, students were making sense of the problem (SMP 1) and using a different model to represent the problem (SMP 4). In fact, using these Mathematical Practices helped them think and act as mathematicians.

USING A VARIETY OF STRATEGIES AND MODELS

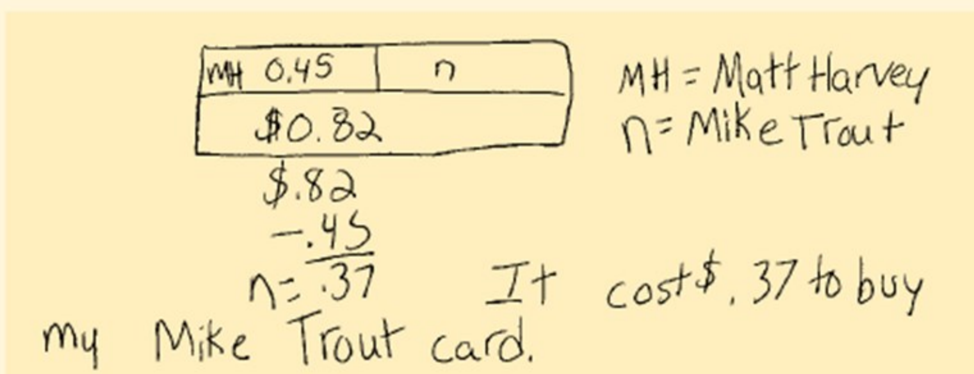
We found that students needed to be able to transfer their learning to other applications and should have opportunities to do so during the learning process. Therefore, students were given a variety of contextual problems to apply these strategies to other situations. After solving and discussing several problems, students worked independently using balance scales, bar diagrams, and mental computation. They also produced contextual problems that would fit a given equation. Skyler created the situation in **figure 3a** for the equation $\$0.45 + n = \$0.82$. He then drew a bar diagram and solved the equation (see **fig. 3b**).

Kerry used a balance scale and reasoned that since \$0.45 and \$0.40 would equal \$0.85, she needed to subtract 3 cents from 40 cents to get her answer of \$0.37 (see **fig. 3c**).

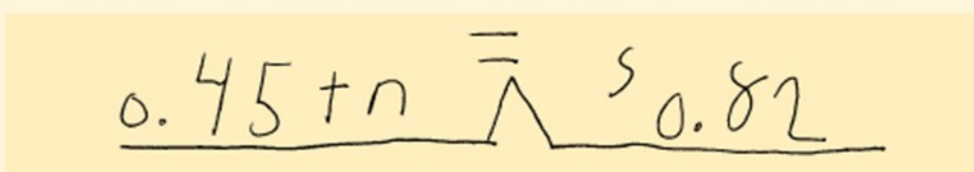
Fig. 3 These examples of students' independent work showed a solidifying of their understanding of solving linear equations.



(a) Skyler's word problem for the equation \$0.45 + *n* = \$0.82



(b) Skyler's bar diagram, used in the solution to his word problem



(c) Kerry's solution to Skyler's problem using a balance scale

We then posed more challenging problems to students, including the problem in **figure 4**. From Dave's work in **figure 4**, we see that he used a bar diagram and an equation with the given information to help solve the problem. He reasoned that since each notebook is $1.00 more than each set of 2 pens, then the colored pencils must be $2.00 more than the cost of the third notebook, which was $2.95. Therefore, the colored pencils cost $4.95. Giving students an opportunity to solve problems in different contexts helped them develop a deeper understanding of equations, the equals symbol (=), and the concept of equality.

Students next moved from contextual to symbolic problems. When working with the scales and bar diagrams, students learned how to solve certain types of equations using compensation strategies. "If I add 3 to a number, I must subtract 3 from another number to maintain balance and equality." They were given the following problems and asked to apply what they learned to find the value of *n* just by reasoning about balance and equality.

$4832 + 197 = n + 200$ $\quad$ $49 + n = 73 + 50$ $\quad$ $23 + n = 14 + 24$

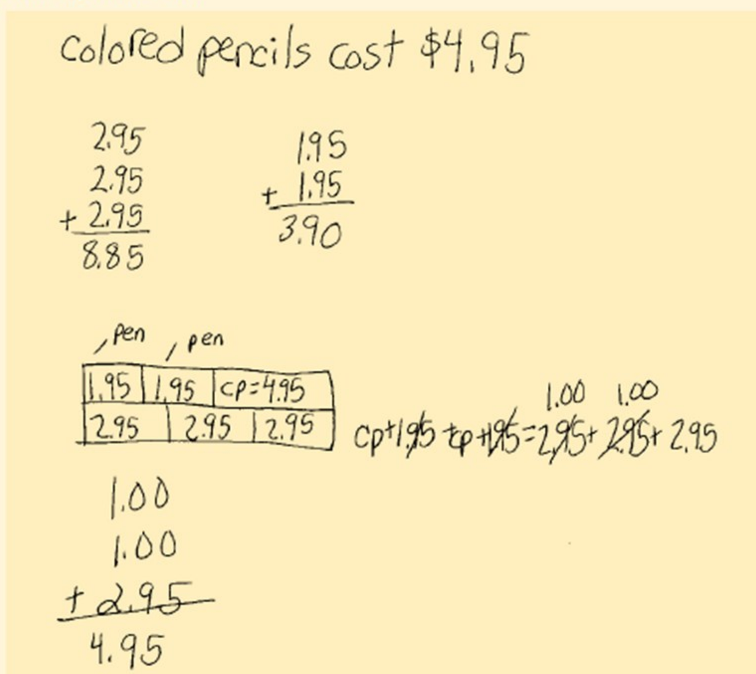
$51 - n = 50 - 25$ $\quad$ $78 + 32 = 80 + n$

This activity helped students build fluency with mental computation, an important skill that is a hallmark of mathematicians. They were also reasoning abstractly and quantitatively (SMP 2). For students who needed more challenge, we asked them to write their own equations, which could be solved using similar compensation strategies, and trade papers with partners to solve.

The equations that students encountered in these initial activities were designed so that they could solve them mentally. Bar diagrams and scales emphasized balancing both sides of an equation to find the solution. As students progressed through the unit, they used tables and flow charts to solve equations. They also learned how to solve using the traditional approach of inverse operations. In so doing, they had a variety of strategies from which to choose to find the solution.

Fig. 4 Dave explored finding the cost of a set of colored pencils.

Notebooks cost $2.95 each, and 2 pens cost $1.95. A set of colored pencils and 4 pens have the same cost as 3 notebooks. How much does a set of colored pencils cost?



WRITING AND SOLVING EQUATIONS WITH TWO VARIABLES

Students then moved to the more challenging task of writing equations with two variables from contextual situations. This new situation required that they focus on the interrelationships among the variables as well as the effects of operations on the variables. In this investigation, students worked on writing such equations.

The activity began by setting the scene at the end of the Silk Road in ancient Egypt and writing an equation that stated the relationship between two animals. For example, when looking at a chart that showed that the trader, Iris, had 20 pigs and 12 horses for sale, students learned that Iris wrote $p - 8 = h$. Her husband, Seth, wrote $h + 8 = p$. Students were then asked to compare and critique different equations to match the same situation and to write other equations of their own to show the relationships among the numbers of the animals being traded.

We deliberately introduced common misconceptions. In one problem, students were given the equation $p + 11 = g$, and they said that this must mean that there were 11 more pigs than goats. In another, Seth saw that Iris had written $2c = h$ and said that she was wrong because they did not have 2 camels and 1 horse but rather 6 camels and 12 horses. Lively discussions ensued as students struggled to make sense of the notation. It is challenging for some students, in particular English language learners, to state these relationships. They often mix up the variables and/or the operations. Exposing students to misconceptions and asking them to critique one another's reasoning (SMP 3) helped them solidify their own understanding.

Fig. 5 Students produced a variety of riddles for systems of equations and solved them in different ways.

1a) Write a riddle that could be represented by the following equations. Explain what the variables stand for.

$f + g = 11$

$f = 5 + g$

f=fossils
g=grapes

I have a total of 11 fossils and grapes. there are 5 more fossils than grapes. How many fossils and grapes are there?

(a) A sample riddle

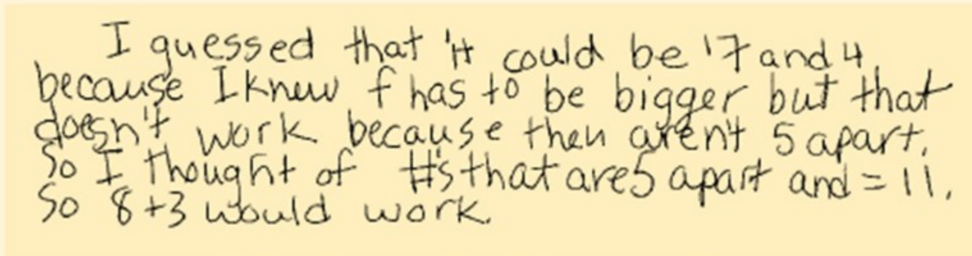
$\overset{10}{f}+\overset{5}{g}=11$		$f=5+g$
$6+5=11$		$6\neq5+5$
$4+7=11$		$4\neq5+7$
$5+6=11$		$5\neq5+6$
$2+9=11$		$2\neq5+9$
$9+2=11$		$9\neq5+2$
$7+4=11$		$7\neq5+4$
$10+1=11$		$10\neq5+1$
$8+3=11$	yes	$8=5+3$

(b) A solution using guess and test

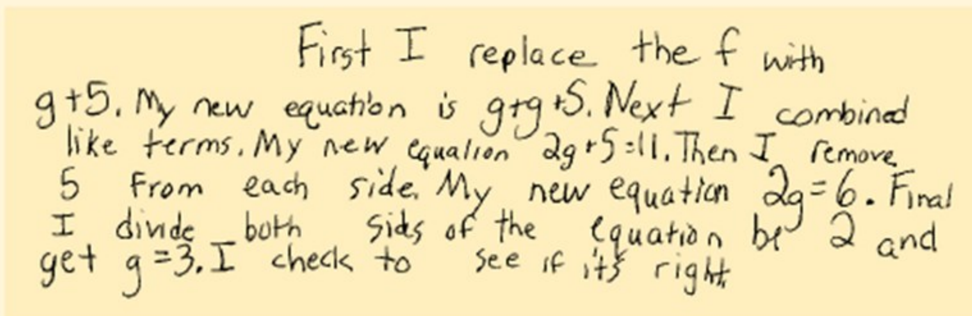
To conclude the unit, students learned how to write and solve sets of two equations (see **fig. 5**) with two unknowns, exploring the relationship between variables in equations. In keeping with our emphasis on sense making, students often began with a guess, test, and refine method (see **fig. 5c**). They then learned how to use substitution to find the solution, which we call the replace, remove, and divide strategy (see **fig. 5d**). Instead of guessing, they replaced a variable in one equation with an equivalent expression found in the second equation. They then removed numbers to isolate the variable. Finally, they divided by the coefficient of the variable to find the value of the variable.

Note that the first student listed all possible combinations of 11 first, using a table to model her thinking, and then found which one worked with the other equation (see **fig. 5b**). A second student explained in words his guess-and-test strategy (see **fig. 5c**), and the third student explained step by step how she used the replace, remove, and divide strategy (see **fig. 5d**). None of the students were randomly guessing but rather they were using methods that logically led to the correct answer. Again, this provided evidence of students thinking and acting like mathematicians.

Fig. 5 Students produced a variety of riddles for systems of equations and solved them in different ways.



(c) Another solution using guess and test



(d) A solution that employed the replace, remove, and divide strategy

How do we know that these types of investigations help students learn? We wanted to find out. We administered open-response pre- and post-unit tests to 305 students. On average, student scores went from 4.63 to 12.23, with 98 percent of students making gains. At the beginning and end of the school year, we also administered an open-response assessment based on CCSSM items used in the Smarter Balanced Assessment Consortium's Mathematics Showcase Materials for grades 6 and 7. Students outperformed a comparison group (with effect sizes at grade 6 = 1.3 and grade 7 = 1.6).

In conclusion, we found that CCSSM-based algebra investigations with a focus on the Standards for Mathematical Practice were successful in helping students develop a much deeper understanding of equality and variables and their relationships in equations. We believe that this will give students a strong foundation on which to build algebraic concepts as they progress through middle school and into high school.

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ABOUT THE AUTHORS

M. Katherine Gavin, mkatherinegavin@aol.com, is a retired associate professor at the University of Connecticut. She is interested in developing mathematics curriculum focused on critical and creative thinking and working with teachers to develop mathematical reasoning and talent in their students. **Linda Jensen Sheffield**, sheffield@nku.edu, is a Regents Professor Emerita of Mathematics Education at Northern Kentucky University. Her interests include the creation and implementation of curricula designed to enhance and support students' mathematical expertise and enjoyment.

Staircases, Towers and Castles

by Melike Kara, Cheryl Eames, Amanda Miller and Annie Chieu

The very nature of algebra concerns the generalization of patterns (Lee 1996). Patterning activities that are geometric in nature can serve as powerful contexts that engage students in algebraic thinking and visually support them in constructing a variety of generalizations and justifications (e.g., Healy and Hoyles 1999; Lannin 2005). In this article, we discuss geometric patterning tasks that engage students with wide-ranging levels of ability, interest, and motivation. This succession of tasks is likely to elicit recursive reasoning strategies to build mathematical sequences on previous terms' values or explicit formulas to determine any value in the sequence. The tasks are increasingly complex in terms of mathematical patterns, numeric computations, and visualization demands.

In this article, we authors—three researchers and a classroom teacher—share how we structured and reflected on our implementation of this task sequence in an Algebra 1 class. We organized students into groups of three and gave each student a role: A recorder would keep track of the group's ideas; a builder would build and help explain the structures; and a reporter would share the group's ideas. We made cubes, graph paper, and calculators available. Each group was given opportunities to present its ideas to the class at various points during the solution process.

For the Algebra 1 class discussed here, we presented two of the tasks over three class periods.

DAY 1

Task 1: Staircases

A group of students are building staircases out of wooden cubes. The 1-step staircase consists of one cube, and the 2-step staircase consists of three cubes stacked (see **fig. 1**). How many cubes will be needed to build a 3-step staircase? A 6-step staircase? A 50-step staircase? An *n*-step staircase?

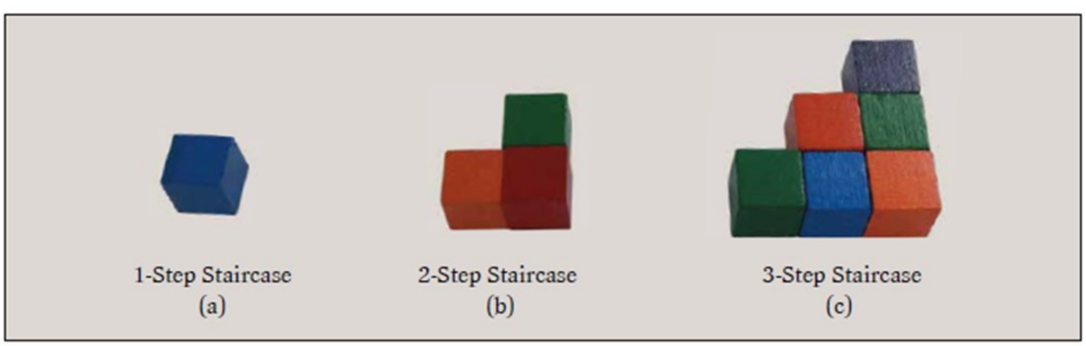

Fig. 1 Cubes are used to build the first three staircases.

While working on the task, students needed to visualize or represent the isosceles right-triangle shaped staircases by building them with cubes or by drawing them. After approximately twenty minutes of exploration time and before any of the groups had generated an explicit expression for the number of cubes needed to build an *n*-step staircase, we invited each group to share its partial solutions. Five of the six groups shared their ideas by drawing and writing on the board. Three of the groups produced a numerical answer for the number of cubes needed to build the 50-step staircase. Two groups came up with an answer of 2500, and one group came up with the correct answer of 1275. (See **figs. 2a–e** for written records of the solutions from five groups.)

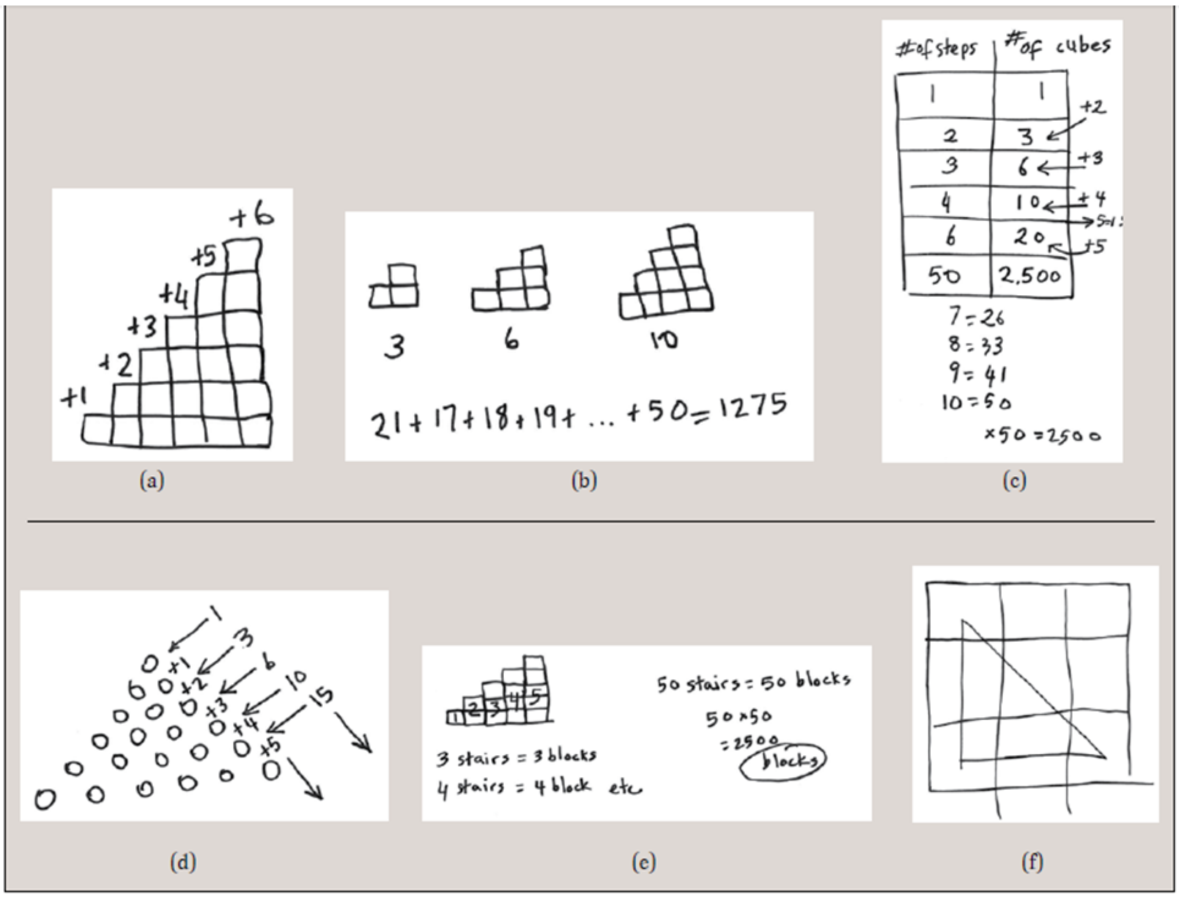

Fig. 2 Students begin solutions to the Staircase task.

Figure 2a represents a group's drawing of the 6-step staircase and the initial phase of what these students noticed in the pattern. Another group attempted to add the number of cubes by using a recursive strategy of summing consecutive integers (see **fig. 2b**). Although this group obtained the correct numeric answer of 1275 for the 50-step staircase, its records had an error (21 + 17 + 18 + 19 + L + 50 = 1275 instead of 21 + 7 + 8 + 9 + L + 50 = 1275). **Figure 2c** is an example of another group's numerical representation of a recursive strategy. Students in this group recorded the number of cubes for each staircase as well as the number of cubes added for each subsequent step. However, they reported the 6-step staircase incorrectly. Later, they multiplied 50 times 50 and concluded that 2500 cubes were needed for the 50-step staircase, an incorrect answer. Another group determined the number of cubes for the first five staircases by drawing the diagram in **figure 2d**. The last group of students thought about the 50-step staircase as a 50 × 50 × 1 wall; they seemed to be thinking about a square number of 50 × 50 (see **fig. 2e**).

One student, Brian (all students' names are pseudonyms), noticed that the 3-step staircase could be transformed into a square with side length of 3 by combining the 2-step and 3-step staircases, but he was not able to use what he noticed to make a generalization about the pattern (see fig. 2f).

This task was a challenge for students. Although they quickly and easily built the 1- through 6-step staircases and drew records to keep track of their ideas, most students struggled to determine the number of cubes that would be needed to build the 50-step staircase. Many students focused on looking for a numerical pattern rather than making use of the geometric shape of the staircases. The classroom teacher's reflection about day 1 reveals the challenge of supporting secondary school Algebra 1 students in generating an explicit formula.

> Overall, students understood that you can determine the number of blocks by adding 1 + 2 + 3 + L + n number of steps, but they didn't know what to do to go beyond that recursive formula. I'm worried that it will be difficult to push them toward an ex plicit formula. Students were asking whether I would tell them the "answer" at the end of today's lesson. . . .

Although students struggled with writing an explicit expression, four of the six groups made significant advances toward a viable solution. After reflecting on the groups' partial solutions, we decided to begin the next class by facilitating a class discussion about two of them—one from a group that relied on recursive thinking (see **fig. 2b**) and another from a group that relied on reasoning about the underlying geometric structure (see **fig. 2f**), which we thought could support an explicit way of thinking about the pattern.

DAY 2

At the beginning of the next class, we reviewed the partial solutions that had been shared the previous day and asked students to discuss the other groups' ideas. All the groups quickly agreed that the number of cubes needed to build the 50 -step staircase could be found by adding all the consecutive integers from 1 to 50. At this point, we returned to Brian's drawing (see **fig. 2f**) to encourage a class discussion aimed at generating an explicitly defined formula and to link this formula to the geometric representation of the pattern.

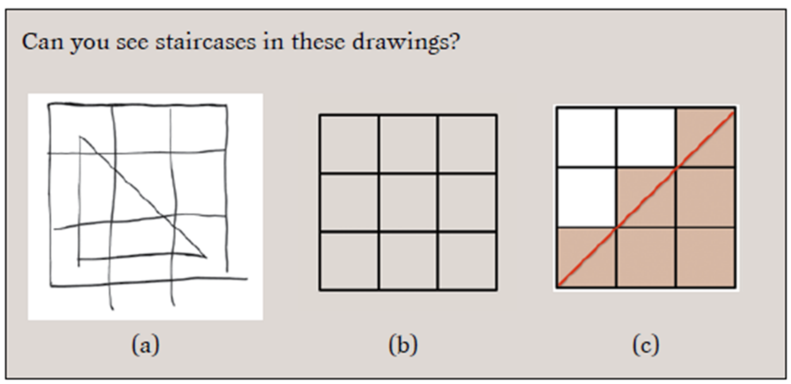

Fig. 3 A shaded diagram related to Brian's work helps clarify his idea.

We created some additional drawings (see **fig. 3**) to help students make sense of Brian's way of thinking about composing two different staircases to form a square. However, the students were not yet able to use what they noticed to make a generalization about the pattern. We next suggested a strategy of composing two of the same staircases to form a rectangle. **Figure 4** illustrates a rotated and translated 3-step staircase on the top of another 3-step staircase, part of the animation that we presented.

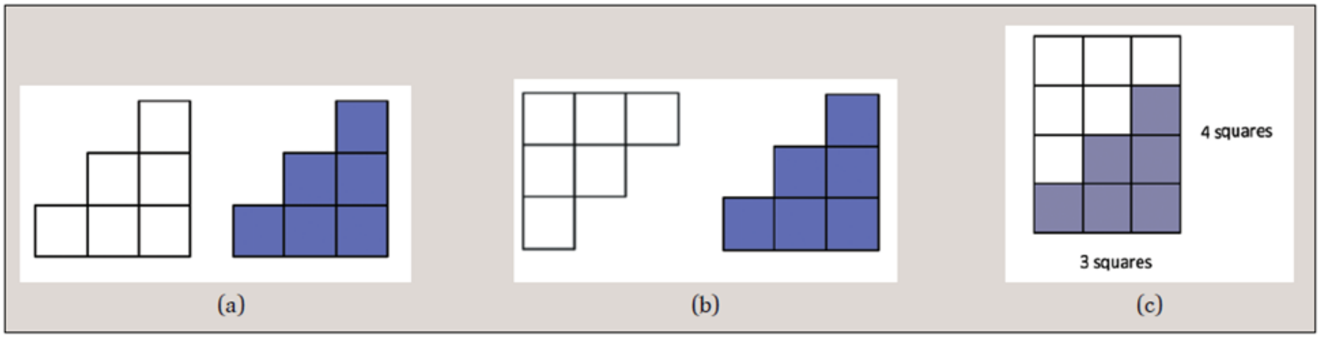

Fig. 4 Snapshots of an animation suggest a method using two identical staircases (**a**), a rotation (**b**), and a translation (**c**).

We asked students to guess how many cubes there are in one 3-step staircase. Students immediately recognized that for two 3-step staircases, there should be 3 times 4 cubes; for one 3-step staircase, that number should be divided into 2. They later applied this geometric approach to larger staircases and talked about whether they could generate a formula based on this approach. Some students came up with the solution for two n-step staircases (by imagining rotating and translating two n-step staircases) to form an $n \times (n + 1)$ wall, providing justification for the explicit expression $n(n + 1)/2$.

For a recursive solution, we introduced the method of adding

$$1 + 2 + 3 + 4 + L + 10 = A \text{ and } 10 + 9 + 8 + 7 + L + 1 = A,$$

resulting in ten 11s for the sum of two As. For one A, this sum should be divided into 2. This method can be generalized to the n-step staircase to produce n pairs, each of which has a sum $(n + 1)$. This method was challenging for most of the students. However, they were involved in the discussion and felt comfortable sharing their thoughts for each step. The questioning style was crucial for each step. Rather than just stating the answer, we wanted them to think about the methods and their generalizability. The reflection of the classroom teacher indicated how this instruction helped some students in the classroom:

> When asked the meaning of dividing by 2, a few students indicated that it was because we were multiplying. I decided to point out that it was related to the picture, and one student stated, "There's two staircases." That student seems to be engaging really well with this activity. She may be a prime example of a student who struggles with the traditional model of the classroom, but she was excelling at this problem-solving activity.
>
> The problem-solving model appeals to many different types of learners. There is the visual representation of pictures, the kinesthetic representation of the blocks, and the auditory representation when the groups share. . . . The atmosphere of the classroom at the end of day 2 had changed significantly since the end of day 1. We can only see what will happen in day 3.

After the students generalized the staircase pattern in both recursive and explicit ways in the first part of day 2, we posed the Skeleton Towers task.

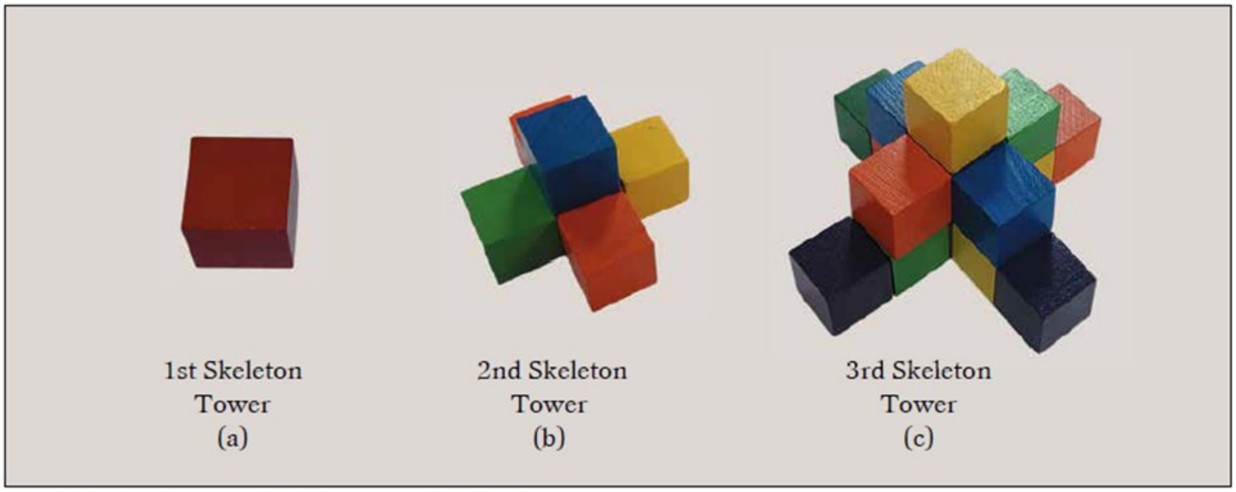

Fig. 5 Skeleton towers extend from a central stack.

Task 2: Skeleton Towers

> A skeleton tower is made up of a stack of cubes with a triangular wing on each of the four lateral faces of the cube. The pictures represent the first three skeleton towers (see fig. 5). How many cubes would be needed to build the 4th skeleton tower? The 5th? The 10th? The nth? How would you describe the pattern verbally?

As with the Staircase task, many students needed to represent the skeleton towers by building them with cubes or by drawing them. We encouraged students to imagine the towers from a bird'seye view and keep track of the number of cubes in each "stack" using the convention displayed in **figure 6**, introducing this as a method that the recorder in another class used to represent some of her group's thinking. At the end of day 2, students were still working on the tasks.

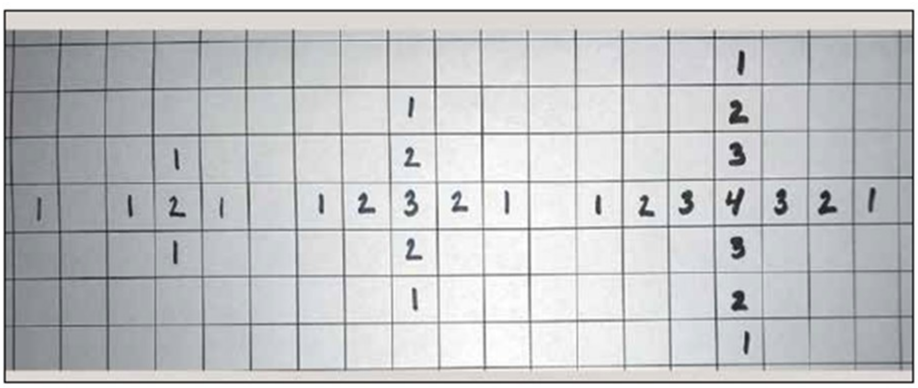

Fig. 6 A bird's-eye-view report allows students to record the number of cubes.

DAY 3

At the beginning of day 3, we asked groups to work again on the Skeleton Towers task and later share their ideas about the tasks. **Figure 7a** represents the solution of one group for the 5th skeleton tower. Two groups generated expressions for the number of cubes in the *n*th skeleton tower by thinking about the staircases as the parts of the tower (see **figs. 7b** and **7c**). They then used what they remembered about the staircase pattern to help them describe this new pattern.

In general, the number of cubes needed to build the *n*th skeleton tower is 4 times the number of cubes in the $(n-1)$-step staircase plus n, or

$$4(n-1)n/2+n=2(n^2-n)+n=2n^2-n,$$

for positive integers, n.

The excerpt below from the teacher's reflection on day 3 reveals the change in the classroom atmosphere as well as classroom norms.

> After finishing the Staircase activity from day 2, the students "knew" how to approach the Skeleton Towers activity. There was great enthusiasm in this environment, and it was apparent that students understood the roles and the norms of the classroom. Although students were willing to share and discuss, there were still students who did not want to present "wrong" information. This may have been instilled within my classroom as well as their previous math experiences. Students may view math problems as having only one answer. In problem solving, there is not one answer or one approach to a situation. It is necessary to create an environment within the classroom where students are not afraid to try things. We have seen over the past three days that even the smallest drawing on the back of a sheet of paper may signify deep processing within the mathematics.

Depending on the challenge level that students can handle, they can be introduced to a third task in the sequence. We did not introduce the third task because of time constraints.

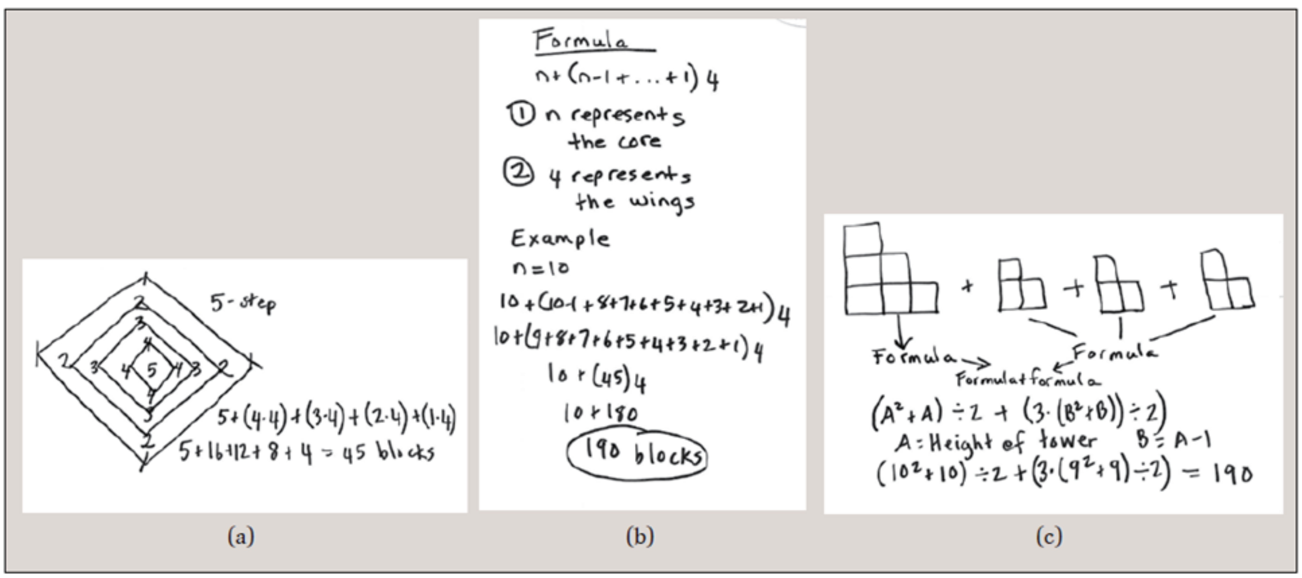

Fig. 7 Students solve the Skeleton Towers task.

Task 3: Skeleton Castles

> A skeleton castle is made up of stacks of cubes that rise on four corners of a square and descend to meet at the middle of each side of the base. Skeleton castles that are 2 cubes high, 3 cubes high, and 4 cubes high are shown (see **fig. 8**). How many cubes would be needed to build a skeleton castle that is 5 cubes high? 6 cubes high? 10 cubes high? n cubes high?

One way to solve this task is to decompose the castle into four stacks of height n, one at each corner of the square base, and eight staircases of height (n –1), two on each of the four sides of the square base. Because each staircase shares the first step, which is a single cube, the first step is double counted four times. In total, the number of cubes needed to build the n-high skeleton castle is

$$4n + 8\left(n-1\right)n/2 - 4 = 4n^2 - 4, \quad \text{for integers } n > 1.$$

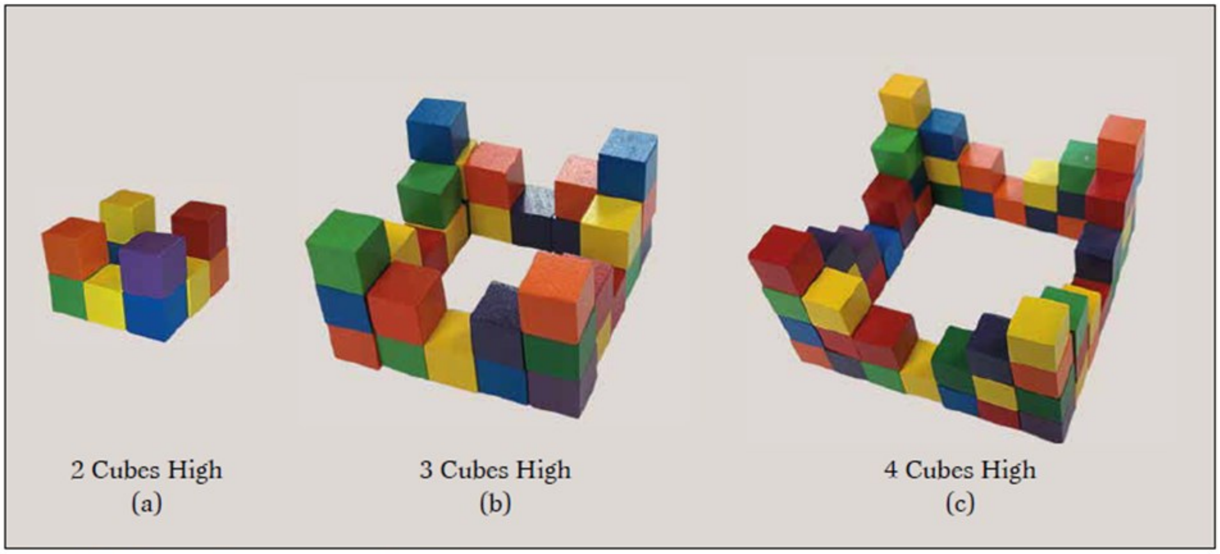

Fig. 8 Skeleton castles have a square footprint.

TASK EXTENSIONS

The tasks described here encourage students to reason algebraically about geometric structures in three-dimensional space. However, each task can be extended to engage students in reasoning about measurement, specifically length and area. The extension prompts include this problem:

> Use your drawing of a bird's-eye view to help you think about the footprint of a staircase (or skeleton tower or castle). What is the area (or perimeter) of the footprint of a 1-step staircase? A 2-step staircase? A 50-step staircase? An n-step staircase?

Further task extensions could be couched in real world contexts involving predicting the height of a growing plant (or fantasy creature), determining the number of rooms in buildings with varying numbers of floors, or constructing borders of gardens of varying shapes and areas. Other scenarios could extend beyond quadratic relationships to include modeling the growth of populations or investments. For example, students could examine past data concerning the number of smartphones sold each year to predict future sales. In each situation, cubes could provide visual support as students quantify the patterns by making sense of geometric structures.

THE LESSON'S PEDAGOGICAL BENEFITS

The teacher's reflections and the students' solutions illustrate the instructional value of this three-day lesson by showing that students gained confidence and sophistication in generalizing and justifying patterns. Day 1 was the most difficult day; however, as the teacher noted on day 2, the classroom atmosphere changed significantly since the end of day 1" as students successfully generalized the Staircases problem pattern in a variety of ways. By day 3, the teacher noted that students "knew how to approach" the Skeleton Towers problem with strategies they refined earlier.

The instructional value of this lesson is further substantiated by its relevance to the Common Core State Standards for Mathematics (CCSSM) treatment of functions (HSF.BF.A.1 and HSF.BF.A.2) and the five CCSSM Standards for Mathematical Practice (SMPs) (CCSSI 2010, pp. 6–8). To support students in making sense of problems and persevering in solving them (SMP 1), we withheld strategy hints and answers. Instead, we allowed time for small-group and whole-class discussions to encourage students to construct viable arguments and critique the reasoning of others (SMP 3). We supported students as they modeled with mathematics (SMP 4) while looking for and making use of structure (SMP 7) by providing different ways to represent the structures (such as building cubes and the bird's-eye-view method of reporting). We found that, for each task, building the pattern helped students reason abstractly and quantitatively (SMP 2). This unique and carefully sequenced set of tasks allowed students to relate numeric (sequences and series), algebraic (explicit expression), and geometric (a building or drawing) structures.

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ABOUT THE AUTHORS

MELIKE KARA, mk4854@nyu.edu, a former mathematics teacher, works as a postdoctoral research associate at New York University. She conducts research on developing learning trajectories for measurement, rational numbers, and geometry. **CHERYL L. EAMES,** ceames@siue.edu, is an assistant professor of mathematics education at Southern Illinois University Edwardsville. Her research is centered on using learning trajectories to study children's conceptions of length, area, and volume measurement. **AMANDA L. MILLER,** almille@ilstu.edu, a former middle school mathematics teacher, teaches mathematics content and methods courses at Illinois State University in Normal. She is interested in students' development of geometric measurement concepts. **ANNIE CHIEU,** annie.chieu@gmail.com, is a high school mathematics teacher in Sachse,

KATM Cecile Beougher Scholarship
FOR ELEMENTARY TEACHERS!!

A scholarship in memory of Cecile Beougher will to be awarded to a practicing Kansas elementary (K-6) teacher for professional development in mathematics, mathematics education, and/or mathematics materials needed in the classroom. This could include attendance at a local, regional, national, state, or online conference/workshop; enrollment fees for course work, and/or math related classroom materials/supplies.

The value of the scholarship upon selection is up to $1000:

- To defray the costs of registration fees, substitute costs, tuition, books etc.,
- For reimbursement of purchase of mathematics materials/supplies for the classroom

An itemized request for funds is required. (for clarity)

REQUIREMENTS:

- Have a continuing contract for the next school year as a practicing Kansas elementary (K-6) teacher.
- Current member of KATM (if you are not a member, you may join by going to www.katm.org. The cost of a one-year membership is $15)

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than June 1 of the current year:

1. A letter from the applicant addressing the following: a reflection on how the conference, workshop, or course will help your teaching, being specific about the when and what of the session, and how you plan to promote mathematics in the future.

2. Two letters of recommendation/support (one from an administrator and one from a colleague).

3. A budget outline of how the scholarship money will be spent.

***Notification of status of the scholarship will be made by July 15 of the current year. Please plan to attend the KATM annual conference to receive your scholarship. Also, please plan to participate in the conference.

SUBMIT MATERIALS TO:

Betsy Wiens , 2201 SE 53rd Street, Topeka, Kansas 66609

Go to www.katm.org for more guidance on this scholarship

Here's how this scholarship helped last year's recipient……..

This fall I was the recipient of the Capital Federal Scholarship offered by KATM. I am the Title Math instructor at Eisenhower Elementary School in Norton, KS. In this position, I go to every classroom from kindergarten through fourth grade giving extra help to students. I also play many math games with my classes to reinforce skills they are currently learning. Many times my colleagues ask to borrow my math supplies so their students can play the games more often. I'm always happy to share, but I thought it would be great if I could make a math manipulative library so that teachers had access to everything they need.

Through this scholarship, a community grant, and an anonymous donor, I have been able to get a great start on making this math library a reality! I've purchased many books, math CDs, dice, dominoes, calculators, games, and all types of manipulatives. I still have about half of my total grant to spend and am still working on a permanent storage area.

It has been so exciting to see the students trying out the new supplies! It is also very rewarding to me to be able to share all these resources with my fellow teachers. We are trying to raise a generation of learners who know that math isn't scary - it's fun! I want to encourage educators to apply for grants and scholarships. A year ago, I would never have dreamed that my math manipulative library would be a reality. It does take a little extra time and effort to apply, but grants can make a big impact in your classroom! Thank you, Capital Federal and KATM for this scholarship. It will continue to give for many years and impact many students.

Capitol Federal Mathematics Teaching Enhancement Scholarship

Capitol Federal Savings and the Kansas Association of Teachers of Mathematics (KATM) have established a scholarship to be awarded to a practicing Kansas (K-12) teacher for the best mathematics teaching enhancement proposal. The scholarship is $1000 to be awarded at the annual KATM conference. The scholarship is competitive with the winning proposal determined by the Executive Council of KATM.

PROPOSAL GUIDELINES:

The winning proposal will be the best plan submitted involving the enhancement of mathematics teaching. Proposals may include, but are not limited to, continuing mathematics education, conference or workshop attendance, or any other improvement of mathematics teaching opportunity. The 1-2 page typed proposal should include

- A complete description of the mathematics teaching opportunity you plan to embark upon.
- An outline of how the funds will be used.

An explanation of how this opportunity will enhance your teaching of mathematics.

REQUIREMENTS:

The successful applicant will meet the following criteria:

- Have a continuing contract for the next school year in a Kansas school.
- Teach mathematics during the current year.

Be present to accept the award at the annual KATM Conference.

APPLICATION:

To be considered for this scholarship, the applicant needs to submit the following no later than **June 1 of the current year.**

- A 1-2 page proposal as described above.

Two letters of recommendation, one from an administrator and one from a teaching colleague.

PLEASE SUBMIT MATERIALS TO:

Betsy Wiens, Phone: (785) 862-9433, 2201 SE 53rd Street, Topeka, Kansas, 66609

Supplies bought with last year's Cecile Beougher scholarship.

KLFA Hears KSBE Vision, ESSA and Assessment Update

•January 6, 2016•

Kansas Learning First Alliance (KLFA) met January 6 at the Kansas Association of School Boards (KASB) building. KLFA celebrated the start of its 17th year working toward its vision to unite the education community to improve the outstanding public education system, pre-K through higher education, and to empower each Kansan to succeed in the diverse, interdependent world of the 21st century. The meeting was shared digitally using Zoom, a teleconferencing system facilitated by Melinda Stanley with KanREN. The KLFA Steering Committee voted to purchase the use of ZOOM for future meetings.

The Legislative Update included the continued concern with educational funding using the Block Grant system. The Efficiency Report and the State of the State will be important toward determining what will happen next. The upcoming election will play a critical role as to what decision makers are at the table.

Beth Fultz, Kansas State Department of Education, shared the proposed state outcomes aligned to the new vision and strategic plan. The proposal included kindergarten readiness, graduation rates, completing a credential or pursuing post-secondary education, individual plans of study with a career emphasis, social/emotional factors, and civic engagement. State assessment information was shared regarding the testing window, caching parameters, the process for registering students, performance tasks, and reporting procedures.

Brad Neuenswander, Deputy Commissioner KSDE, shared key points from the recently passed Every Student Succeeds Act (ESSA). ESSA maintains annual assessments and authorizes innovative assessment pilots and provides states with increased flexibility to design school accountability systems, school interventions, and student supports. Further, it allows states flexibility to work with local stakeholders to develop educator evaluation and support systems and increases state and local flexibility in the use of federal funds.

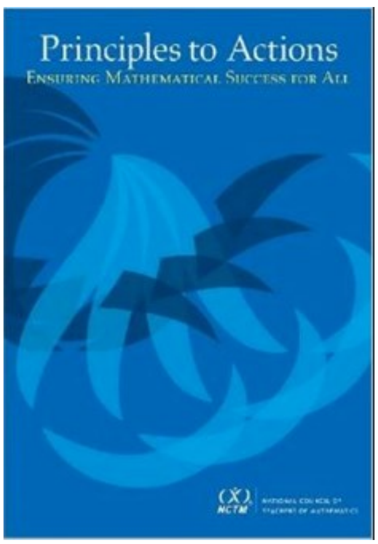
Members continued working in one of the three focus areas: **Professional Learning**, **Student Success** and **Community Engagement**. **Professional Learning** is working with Learning Forward Kansas (LFKS) to develop a videos series that demonstrates professional learning that changes practice, plus creating several resources/tools that focus on the vocabulary of ESSA, KESA, and other state initiatives. **Student Success** is creating resources/tools to support districts focused on Student Success and Kansans Can, especially Individual Plans of Study. **Community Engagement** is also creating resources/tools available for Kansas educators.

The next KLFA meeting is **Thursday, April 14, at the Kansas NEA building**. Please visit our website for more information about KLFA and our work.

NCTM Update

My name is Stacey Bell and I am pleased to be the NCTM Rep for KATM. As I stated in our last Bulletin, NCTM has a new website design and has been focusing on developing its Affiliate Site for its members. As an affiliate of NCTM, KATM is able to now post our upcoming events on this new site for neighboring states to see. And likewise, we are able to see what other affiliates are doing around us. You should check it out at http://www.nctm.org/affiliates/

In other news, NCTM has published a new book, *Principles to Actions: Ensuring Mathematical Success for All.* Below is NCTM's description of the book found at http://www.nctm.org/Store/Products/Principles-to-Actions--Ensuring-Mathematical-Success-for-All/



The widespread adoption of college- and career-readiness standards, including the Common Core State Standards for Mathematics, presents a historic opportunity to improve mathematics education.

What will it take to turn this opportunity into reality in every classroom, school, and district?

Continuing its tradition of mathematics education leadership, NCTM has defined and described the principles and actions, including specific teaching practices, that are essential for a high-quality mathematics education for all students.

Principles to Actions: Ensuring Mathematical Success for All offers guidance to teachers, specialists, coaches, administrators, policymakers, and parents:

- Builds on the Principles articulated in Principles and Standards for School Mathematics to present six updated Guiding Principles for School Mathematics
- Supports the first Guiding Principle, Teaching and Learning, with eight essential, research-based Mathematics Teaching Practices

Details the five remaining Principles—the Essential Elements that support Teaching and Learning as embodied in the Mathematics Teaching Practices

KATM Spring Election Nominations

VP High School—Amber Hauptman teaches at Washburn Rural High School. She has been actively involved in curriculum planning on the district level. She also participated in the setting cut scores for state assessments in summer 2015.

VP Elementary—Hello! My name is Amy Johnston and I am running for the KATM Vice President – Elementary position. I currently teach 2nd grade at Auburn Elementary where I also serve on the district math committee. I have also taught Preschool and Kindergarten both at Auburn. I graduated from Emporia State University with a degree in Elementary Education and an emphasis in Early Childhood. My master's degree in Instructional Design is also from ESU. I am currently pursuing my National Board Certification. Math has always been a favorite subject. I have especially enjoyed the transition to Common Core as it has encouraged me to spend more time working on number sense and allowing more time to talk about different strategies rather than just one "right way". I firmly believe that a love of math can begin in the primary classroom. It is my goal for all my students to know that math will be challenging at times, but as long as they persevere, we will grow together.

President Elect—Stacey Ryan During her 15 years as an educator, Stacey Ryan has taught 6th-8th grade math at Andover Middle School. Her passion is facilitating real-world applications and projects to make math relevant and meaningful for her students. She wants all students to be confident in math, develop leadership skills, and collaborate effectively with one another as well as professionals who use math in their jobs. As a classroom teacher, Stacey has been recognized as a Microsoft Innovative Educator Expert and Regional Lead, State Finalist for the Presidential Award for Excellence in Math and Science Teaching, Horizon Award recipient, and a WBJ 40 Under 40 Honoree. She is a Skype in the Classroom Master Teacher, WEB Leader Coordinator, Fishtree Ambassador, and Remind Connected Educator. Stacey wants students to develop skills to experience success not only in her classroom, but in life.

CALL FOR SUBMISSIONS

Your chance to publish and share your best ideas!

The KATM Bulletin needs submissions from K-12 teachers highlighting the mathematical practices listed above. Submissions could be any of the following:

- ◊ Lesson plans
- ◊ Classroom management tips
- ◊ Books reviews
- ◊ Classroom games
- ◊ Reviews of recently adopted resources
- ◊ Good problems for classroom use
- ◊

Email your submissions to our Bulletin editor: wilcojen@usd437.net

Do you like what you find in this Bulletin? Would you like to receive more Bulletins, as well as other benefits?

Consider becoming a member of KATM.

For just $15 a year, you can become a member of KATM and have the Bulletin e-mailed to you as soon as it becomes available. KATM publishes 4 Bulletins a year. In addition, as a KATM member, you can apply for two different $1000 scholarship.

Current members—-refer three new members and you get one free year of membership!

Join us today!!! Complete the form below and send it with your check payable to

KATM to:
Margie Hill
KATM-Membership
15735 Antioch Road
Overland Park, Kansas 66221

Name________________________________

Address______________________________

City_________________________________

State________________________________

Zip_________________________________

Home Phone___________________________

HOME or PERSONAL EMAIL:

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Are you a member of NCTM? Yes___ No___
Position: (Cirlce only one)

Parent
Teacher:: Level(s)________
Dept. Chair
Supervisor
Other

Referred by: ________________________________

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KANSAS ASSOCIATION MEMBERSHIPS
Individual Membership: $15/yr. ___
Three Years: $40 ___
Student Membership: $ 5/yr. ___
Institutional Membership: $25/yr. ___
Retired Teacher Membership: $ 5/yr. ___
First Year Teacher Membership:$5/yr. ___
Spousal Membership: $ 5/yr. ___
(open to spouses of current members who hold a regular Individual Membership in KATM)

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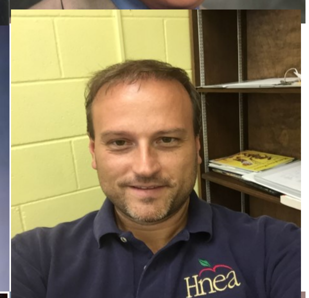

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marghill at @ ku.edu

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albf2201 at aol.com

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epeyser at usd259.net

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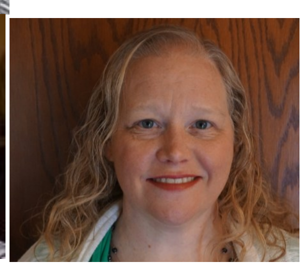
mfast at ksde.org



Bulletin Editor: Jenny Wilcox, 7th grade teacher, Washburn Rural Middle School,
wilcojen at usd437.net

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Zone 3 Coordinator:

Whitney Czajkowski-Farrell, 7th Grade teacher, Shawnee Heights Middle School,

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Jeanett.moore at usd480.net

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